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EUCLID'S ELEMENTS.

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THE

FIRST BOOK

OF

EUCLID'S ELEMENTS,

SIMPLIFIED, EXPLAINED, AND ILLUSTRATED,
FOR THE USE OF BEGINNERS;

WITH AN APPENDIX,

CONTAINING A FEW PROPOSITIONS, OF WHICH THE SOLUTION
IS LEFT TO THE INGENUITY OF THE STUDENT.

BY THE REV.
WILLIAM TROLLOPE, A.M.

LONDON:
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MDCCCXLVII.

BIBLIOTHECA · BODLEIANA

THE FIRST BOOK OF EUCLID'S ELEMENTS.

INTRODUCTORY NOTE.

Mathematics, from the Greek μάθημα, signifies *science;* and it is so called, as it were *par excellence*, because it is the only branch of knowledge which admits of a rigorous demonstration. It treats of whatsoever has relation to number or magnitude; and Geometry is that particular division of it which investigates the mensuration and properties of lines and angles, of surfaces and solids.

Geometry is said to have originated with the Egyptians in the necessity of remeasuring the lands of which the periodical inundations of the Nile continually effaced the boundaries. Hence the science derived its name from the Greek words γῆν and μετρεῖν, or γεωμετρεῖν. Thales of Miletus carried it into Greece, where, from the rude state in which it first existed, it was gradually enriched by the discoveries and demonstrations of Pythagoras, Anaxagoras, Plato, Leodamas, and others, who extended its application to astronomical and other philosophical purposes. Among these, the most celebrated was Euclid, a native of Alexandria, who flourished in the reign of Ptolemy Lagus, about B.C. 280. He was, in fact, the first who reduced Geometry into the form of a science, by arranging the discoveries of preceding mathematicians, and the results of his own labours, with systematic regularity and order. Some writers have improperly confounded him with Euclid of Megara, who lived more than a century before him.

As the theoretical deductions of Geometry depend almost entirely upon the results of close and accurate reasoning, its study is eminently calculated to fix the attention, to exercise the memory, and to enlarge the powers of the mind. In its practical application, on the other hand, it enters into all the various occupations of mankind, from the work of the humblest artificer, to the sublime investigations of the astronomer engaged in the solution of the most complex problem connected with the system of the universe. Euclid's "Elements" have ever been regarded as the groundwork of the science, and it is scarcely probable that they will be superseded by a system of superior excellence, so far as the subject of Geometry is concerned. Of the fifteen books, the first six treat of *plane*, and the eleventh and twelfth of *solid*, Geometry; the rest being chiefly concerned with the principles of numbers, which have been more fully and fitly investigated by modern arithmeticians. The first book is mainly occupied with the relations subsisting between the sides and angles of triangles, and the conditions of equality which obtain between two or more of these, the simplest of all geometrical figures. Some properties of parallelograms are also investigated in connection with those of triangles; together with those particulars respecting perpendicular and parallel straight lines, which are essential to the proof of each succeeding proposition.

DEFINITIONS.

1. A *Point* is that which has no parts, and which has no magnitude.

Definitions are concise explanations of certain elementary principles upon which a science depends; and those which are prefixed to this book are, for the most part, peculiarly simple, exact, and perspicuous. To the first, however, it is objected that it is entirely negative, and that it is not convertible; for though a point is without extension, every thing unextended—as, for example, an instant of time—is not a point. Other definitions have been proposed, of which that given by Professor Playfair is as good as any; viz., that *a point is that which has position, but not magnitude.* The fact is, that abstract ideas, from which the first principles of Geometry take their rise, scarcely admit of strict definition, marking some specific distinction; but Euclid has arranged the few elementary notions, so to speak, upon which the science rests, with such admirable nicety, that their import is clearly understood, though it may not have been perfectly developed.

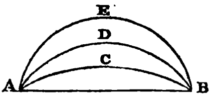
2. A *line* is length without breadth.

3. The *extremities* of a line are points.

This is not so much a definition, as an inference from the two which precede it. As a line has no breadth, its extremity can have none; and if it had length, it would be a continuation of the line, not a termination. Having, therefore, position without magnitude, it is a point. Lines which have no termination—as circles and other curvilinear figures—are not of course taken into the account.

4. A *straight line* is that which lies evenly between its extreme points.

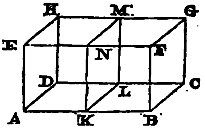
In other words, it is the shortest of all lines which have the same extreme points, according to the definition of Archimedes. Thus AB is evidently shorter than ACB, ADB, AEB, or any other line which does not lie *evenly* between A and B.



5. A *superficies* is that which has only length and breadth.

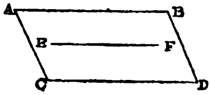
6. The *extremities* of a superficies are lines.

The remarks which were made under the third definition will, *mutatis mutandis*, equally apply here; spherical or curved superficies or surfaces (*super* and *facies*), which have no boundaries, not being taken into consideration. It will also materially conduce to a right understanding of the definitions of a point, a line, and a superficies, if we consider the nature of a solid, in which they all originate, and in which exist the several dimensions of length, breadth, and thickness. For if the solid ABCDEFGH is considered to be composed of the two *laminæ* or smaller solids AM, BM, the superficies KLMN is the common boundary of these solids; and it cannot be a part of the thickness of the solid AM, because if this solid be removed, it still remains as the boundary of the solid BM. For a like reason, it cannot be part of the thickness of the solid BM. Therefore the superficies KLMN has no thickness, but only length and breadth. In the same manner the line KL is the common boundary of the two superficies AL, KC; and, as it partakes of the breadth of neither, it has length only without breadth. So again the point K is the common extremity of two lines AK, BK; and, partaking of the length of neither, has neither length, breadth, nor thickness, but simply position without magnitude. Hence the common intersection of one line with another are points; of one superficies with another, lines; and of one solid with another, surfaces. It is clear also that straight lines can intersect each other in one point only. In practice the mathematical ideas of a point, line, and surface no longer exist, since even a point must have some magnitude, nor is a line or surface absolutely without breadth or thickness. Still it is only necessary to regard these several dimensions as continually diminishing to the verge of evanescence, and the theory and practice are easily reconciled.

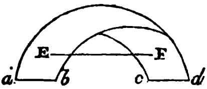


7. *A plane superficies* is that in which any two points being taken, the straight line between them lies wholly in that superficies.

A plane superficies is more frequently called simply a *plane;* and the derivation of the word is from the Latin *planus*, signifying *flat*, or *level*. In the superficies ABCD,



coinciding with the plane of the paper, the line joining any two points, E and F, lies clearly within the superficies; but in *abcd*, which stands out of the plane of the paper, the line joining E and F does not so.



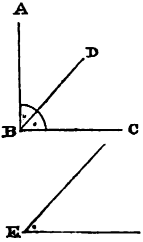
8. "A *plane angle* is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction."

This definition comprehends the angles formed by the meeting of curved, as well as straight lines; but it is with the latter alone, of which the next definition treats in particular, that the Elements of Euclid are concerned. The word *angle* is manifestly derived from the Latin *angulus*, a corner: but an angle is a very distinct thing from a corner, and the student should be careful to obtain an accurate notion of their difference. The one is an external *edge*, the other an internal *inclination*.

9. A *plane rectilineal angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

N. B.—When several angles meet at a point, each of them is indicated by three letters, of which the one at the *vertex* of the angle is read between the other two. Thus the angles ABC, ABD, DBC are formed at B by the lines AB, BC; AB, BD; and BD, BC; respectively. A single letter is sufficient, when there is only one angle at a point; as the angle E.

It is clear that angles are altogether unaffected by the length of the lines, by the meeting of which they are formed. See the Observation after Definition 18.



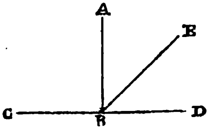
10. When a straight line, standing on another straight line, makes the adjacent an-

gles equal to one another, each of the angles is called a *right angle;* and the straight line which stands on the other is called a *perpendicular* to it.

11. An *obtuse angle* is that which is greater than a right angle.

12. An *acute angle* is that which is less than a right angle.

Thus the angles ABC, ABD are each of them *right angles;* EBC is an *obtuse*, and EBD an *acute* angle; and the line AB is a *perpendicular* to CD.



13. "A *term* or *boundary* is the extremity of any thing."

14. A *figure* is that which is enclosed by one or more boundaries.

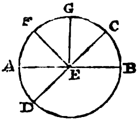
The quantity of space contained in any figure is called its *area.*

15. A *circle* is a plane figure contained by one line, which is called the *circumference;* and is such that all straight lines, drawn from a certain point within the figure to the circumference, are equal to one another:—

16. And this point is called the *centre* of the circle.

17. A *diameter* of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

18. A *semicircle* is the figure contained by a diameter, and the part of the circumference cut off by the diameter.



The subjoined figure is a *Circle*, of which the curved line AFGCBD is the *circumference;* the point E, the *centre*, from which all lines EA, EF, EC, &c., drawn to the circumference, are equal. In fact the circle may be supposed to be generated by the revolution of any one of these lines, as EA, round the fixed point E, until it returns to the point from which it began to revolve. Each of these equal lines is called a *radius* or *ray;* and is evidently one-half of a diameter. Any diameter, as AB, not only divides the circle into two *semicircles*, with reference to the equal semi-circumferences ACB, ADB; but the included areas are also manifestly equal.

It may be well to observe that, for geographical and other purposes, the circumferences are supposed to be divided into 360 equal parts, called *degrees;* and, again, each degree into 60 *minutes*, each minute into 60 *seconds;* and so on. These divisions are symbolically expressed thus: 46° 4′ 13″. Now, in the revolution of the radius AE round the point E, through one-fourth of the circumference AG, the angle at E will gradually have encreased, till it is subdivided or measured by an *arc*—so any portion of the circumference is called—of 90°. Again, in the passage from G to B, another arc of 90° is described, and the angle GEB is equal to the angle AEG. Each is therefore a right angle. Angles are said to be of as many degrees as the arcs upon which they stand; and it is clear that, whatever the radius of the circle, the inclination of an angle of any given number of degrees is always the same; so that an angle is not encreased or diminished by the lengthening or shortening of the lines by which it is formed.

19. *Rectilineal* figures are those which are contained by *straight* lines.

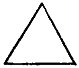
20. *Trilateral* figures, or *triangles,* by *three* straight lines.

21. *Quadrilateral* by *four* straight lines.

22. *Multilateral* figures, or *polygons,* by *more than four* straight lines.

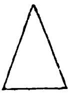
As all rectilineal figures have the same number of angles as sides, they may be named indifferently from either; or sometimes from one, and sometimes the other. Thus, three-sided figures are called *triangles*, from the Latin *tres* and *angulus;* and many-sided figures, *polygons*, from the Greek words πολὺς, *many*, and γωνία, *an angle*.

23. Of three-sided figures, an *equilateral* triangle is that which has three *equal* sides.



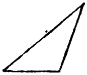
24. An *isosceles* triangle is that which has only two sides equal.

The word *isosceles* is derived from the Greek; and signifies *having equal legs*, that is *sides*, from ἴσος, *equal*, and σκέλος, *a leg*. The base of such a triangle may be either shorter or longer than either of the other sides.

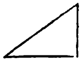


25. A *scalene* triangle is that which has three unequal sides.

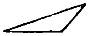
The word *scalene* is also of Greek origin, signifying *unequal*.



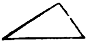
26. A *right-angled* triangle is that which has a right angle.



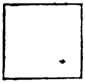
27. An *obtuse-angled* triangle is that which has an obtuse angle.



28. An *acute-angled* triangle is that which has *three* acute angles.

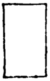


29. Of four-sided figures, a *square* is that which has all its sides equal, and all its angles right angles.

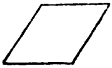


30. An *oblong* is that which has all its angles right angles, but has not all its sides equal.

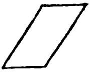
Called also a *rectangle*.



31. A *rhombus* is that which has all its sides equal, but its angles are not right angles.



32. A *rhomboid* is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.



These last four figures come also under the more general definition of a *parallelogram;* which is given under Prop. XXXIV.

33. All other four-sided figures besides these, are called *Trapeziums.*

Trapezium is a Greek word, signifying *a table.*

34. *Parallel* straight lines are such as are in the same plane; and which, being produced ever so far both ways, do not meet.

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line: and

3. That a circle may be described from any centre, at any distance from that centre.

A *postulate* (from *postulare*) is a *demand* so reasonable that it cannot fail to be granted; or, in other words, an assumption of which the grounds are so apparent, as to be admitted without dispute.—*To produce* is a technical term, signifying *to prolong*, or *extend*, from the Latin *producere.*

It appears from these postulates that the instruments which Euclid places in the hand of the student, are, a *straight scale*, and a *pair of compasses.* The instruments employed in practical geometry involve principles investigated in the Elements, which will be noticed as occasions arise.

AXIOMS.

1. Things which are equal to the same thing, are equal to one another.

Also, things which are equal to equal things, are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Things which are double of the same, are equal to one another.

7. Things which are halves of the same, are equal to one another.

8. Magnitudes which *coincide* with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot inclose a space.

In order to enclose a space, two straight lines must have more than one common intersection, which has been already seen to be impossible (Note on Definition 6).

11. All right angles are equal to one another. (See Definition 10.)

12. If a straight line meets two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines, being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

An *Axiom* is a self-evident truth, and is so called from its being worthy (ἄξιος) of universal acknowledgment. The above statement is not an Axiom, but the enunciation of a Theorem, easily capable of proof, but certainly requiring one. A proof is appended to the 29th Proposition.

N. B.—The "Elements" of Euclid are set forth in a series of *Propositions*, of which there are two forms, viz.:—the *Problem* (πρόβλημα), and the *Theorem* (θεώρημα), indicating respectively a question *proposed for resolution*, and *a fact submitted to demonstration*. The results of the operation, in the former case, are marked by the letters Q. E. F. (*quod erat faciendum*), and in the latter, by Q. E. D. (*quod erat demonstrandum.*) Each Proposition consists of four parts:—1. The *general enunciation*, or statement of the object to be effected or proved. 2. The *enunciation* repeated with reference to a *particular figure*. 3. The *construction*, by means of certain lines drawn, or points determined, in order to complete the solution or proof. 4. The *demonstration*, either by a *direct* process of reasoning, or, *indirectly*, by proving a contrary supposition to be *absurd*.—The enunciation of a *Problem* proposes certain data, or premises, from whence to deduce a required result: that of a *Theorem* is divided into the *hypothesis*, or fact assumed to be true, and the *conclusion* built thereon, and requiring demonstration. A *corollary* is an easy inference from the whole or part of a preceding demonstration. These several parts will be distinguished in each Proposition. In the *General Enunciations*, the *data* and *hypotheses* will be printed in *Roman* characters, the *requisition* or *conclusion* in Italics: in the *Particular Enunciation* this distinction will be indicated by the words *Let* and *Then;* and in the *Demonstrations* the successive steps will be numbered, and their object, in less obvious cases, pointed out. Some few propositions require no construction; and in others the construction is mixed up with the demonstration. Partly with a view to conciseness of expression, and partly as a means of rendering the connection, before each successive steps in the operation, more perspicuous and tangible, the following symbols have been employed to mark the relation between the several magnitudes, which come under consideration in the propositions of Euclid, and in those which have been added by way of illustration.

The sign +, placed between two magnitudes, indicates their *addition.*

−, placed between two magnitudes, shows the remainder, when the latter is *subtracted* from the former.

= denotes equality.

> signifies *greater;* or, placed between two magnitudes, shows that the former is *greater than* the latter.

< signifies *less;* or shows that the former of two magnitudes is *less than* the latter.

∥ signifies *parallel,* and $\square^{m}$ *parallelogram.*

⊥ „ *perpendicular.*

∠ „ *angle,* and rt. ∠ *right angle.*

⊙ „ *circle.*

$\bigcirc^{ce}$ „ *circumference.*

△ „ *triangle.*

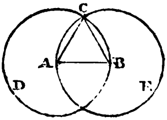
∴ „ *therefore.*

The square of any magnitude, as the line AB, is marked thus, AB^2; and the abbreviations Def., Post., Ax., Prop., Cor., Hyp., Const. imply *Definition, Postulate, Axiom, Proposition, Corollary, Hypothesis,* and *Construction.* Also, rad. is for *radius,* diam. for *diameter,* st. for *straight,* cr. for *centre,* pt. for *point,* gn. for *given,* &c. &c. &c.

PROPOSITION I. PROBLEM.

GENERAL ENUNCIATION.—*To describe an equilateral triangle* upon a given finite straight line.

PARTICULAR ENUNCIATION.—Let AB be the gn. st. line; *then* it is required to describe an equilateral △ upon it.



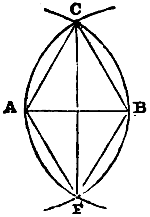
CONSTRUCTION.—From the cr. A, with rad. AB, describe ⊙ BCD; from the cr. B, with rad. BA, describe ⊙ ACE (Post. 3); from the pt. C, where the $\odot^s$ intersect, draw CA, CB to the pts. A and B (Post. 1); *then* ABC is an equilateral △.

DEMONSTRATION.—1. Because the pt. A is the cr. of ⊙ BCD, ∴ AC = AB (Def. 15).

2. Because the pt. B is the cr. of ⊙ ACE, ∴ BC = BA (Def. 15).

3. Hence AC and BC are each = to AB.

4. But things which = the same thing, are = to one another (Ax. 1); ∴ AC = AB = BC: *i. e.* the △ ABC is equilateral; and it has been described upon the gn. st. line AB.—Q. E. F.



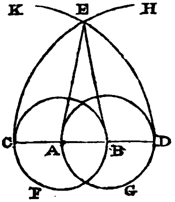
It is clear that, upon the other side of AB, another equilateral △ ABF, may in like manner be described; and if CF be joined, the $\triangle^s$ ACF, BCF are isosceles, having the base, or under side, longer than either of the two equal sides AC, AF, and BC, BF. A more general proposition than the following will be seen under Prop. III. See also Prop. XXII.

This Problem is practically applied in *Fortification*.

PROP. A. PROB.

Gen. Enun.—*To describe an isosceles triangle* upon a given finite straight line, which shall be less than either of the equal sides.

Part. Enun.—Let AB be the gn. st. line; *then* it is required to describe upon it an isosc. △, of which the two = sides shall each be greater than AB.



Const.—From crs. A and B, with radii AB and BA, describe $\odot^s$ BCF, ADG (Post. 3); produce AB both ways to C and D (Post. 2); from crs. A and B, with radii AD and BC, describe $\odot^s$ DEK, CEH (Post. 3); and from the pt. E, where these $\odot^s$ intersect, draw EA, EB to pts. A and B (Post. 1); *then* ABE is the isosc. △ required.

Demonst.—1. To prove that AD = BC.

By property of the $\odot$, AC = (AB =) BD (Def. 15). To each of these equals add AB; ∴ AD = BC (Ax. 2).

2. To prove that AE = BE.

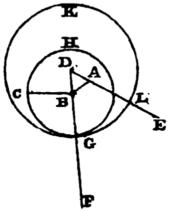
Again, by property of $\odot$, AE = AD, and BE = BC (Def. 15); ∴ AE = BE (Ax. 1).

3. Hence the △ ABE is isosceles, and it is described upon the gn. line AB, which is > AE or BE.—Q. E. F.

PROP. II. PROB.

Gen. Enun.—From a given point *to draw a straight line equal to a given straight line.*

Part Enun.—Let A be the gn. pt., BC the gn. st. line; *then* it is required to draw from the pt. A, a st. line = BC.



Const.—From the pt. A to B, draw the st. line AB (Post. 1). Upon AB describe the equilateral △ DAB. (Prop. I.) Produce the st. lines DA, DB, to E and F (Post. 2). From cr. B, with rad. BC, describe the ⊙ CGH. From cr. D, with rad. DG, describe the ⊙ GKL (Post. 3). *Then* AL is the line required.

DEMONST.—1. By the property of the ⊙, BC = BG (Def. 15).

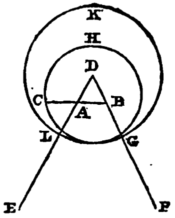
2. To prove that BG = AL.

For same reason, DG = DL (Def. 15). Now, by the construction, DB = DA; ∴ by subtraction, BG = AL (Ax. 3).

3. ∴ also AL = BC (Ax. 1).

Wherefore, from the gn. pt. A, a st. line AL has been drawn equal to the gn. st. line BC.—Q. E. F.

It should be continually borne in mind by the student, that the propositions of Euclid are of universal application; and that, although the figure may illustrate only a particular case, the demonstration is general. Thus, in the present problem, the gn. pt. might be placed either above or below the gn. line, or anywhere upon the line, whether at the extremity or otherwise; or, again, beyond the line, so that AB, when joined, would be in the same st. line with BC. When the pt. is placed at either extremity of the line, as at B, the problem is readily solved by describing one ⊙ GCH only; and BG is the line required. The problem may also be solved by producing BD upwards, instead of downwards, through D to the ○^{ce}; by describing a ⊙ from the cr. D, with the produced line as rad.; and producing AD through D to the ○^{ce}. In this case, the required line is the *sum* of the rad. of the ⊙, whose cr. is D, and the side of the △;—not their difference. Since then the gn. pt. may be joined to either extremity of the gn. line; since the = lateral △ may be described on either side of the line so joined; and since the problem may be solved in two ways with each △; it is clear that there are 8 different lines which may be drawn to meet the required conditions. The student will find it a useful

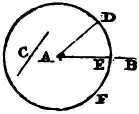


exercise to work the several cases, of which the figure, wherein the point is placed upon the gn. line, is here subjoined; and the construction and demonstration will be found to be precisely the same, word for word, and letter for letter, as that which Euclid has given.

It may be supposed that this problem would be sufficiently resolved by measuring with the compasses the distance BC, and setting off the same distance from A. *Practically* indeed this may be the case, but the *mathematical demonstration* would be wanting; and the connection between this proposition, and those which depend upon it, would not be accurately established. The same remark will apply to many propositions, of which the truth would readily be admitted without proof. A link in the scientific chain is thereby broken; and the system is of course incomplete.

PROP. III. PROB.

GEN. ENUN.—From the greater of two given straight lines *to cut off a part equal to the less.*



PART. ENUN.—Let AB and C be the two gn. st. lines, of which AB is > C; *then* it is required to cut off from AB a part = C.

CONST.—From the pt. A, draw AD = to C (Prop. II.); and from the cr. A, with rad. AD, describe the ⊙ DEF (Post. 3); *then* AE is the part required.

DEMONST.—For by the property of the ⊙, AE = AD (Def. 15); and by the construction AD = C; ∴ also AE = C (Ax. 1). Hence from AB, the > of two gn. st. lines, a part AE has been cut off, which is = to C, the <.—Q. E. F.

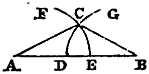
Of this Proposition there are also several cases, inasmuch as the given lines need not be separated; but the shorter may be attached either to the extremity of the other, when the application of Prop. II. becomes unnecessary; or the two lines may meet or intersect each other somewhere between the extreme points; or, again, the less may be a part of the greater. The student will do well to prove each case, but it is superfluous to insert them here. We are now, however, in a condition to give the following general solution of Prop. A. The principle is carried yet further in Prop. XXII.

PROP. B. PROB.

Gen. Enun.—*To describe an isosceles* △ on a given finite straight line.

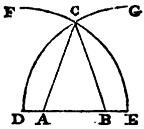
Part. Enun.—Let AB be the gn. st. line; *then* it is required to describe an isosc. △ upon it.

Fig. 1.



Const.—In AB, produced if necessary, take any pt. D, and make AE = BD (Prop. III). From cr. A, with rad. AE, describe ⊙ ECF. From cr. B, with rad. BD, describe ⊙ DCG (Post. 3). From point of intersection C, join AC, BC (Post. 1). *Then* ABC is the △ required.

Fig. 2.

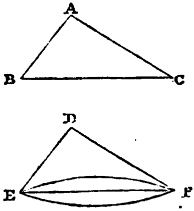


Demonst.—By the property of the ⊙, AE = AC, and BD = BC (Def. 15). But, by construction AE = BD; ∴ also AC = BC (Ax. 1). Hence the △ ABC is isosceles, and it has been described upon the gn. straight line AB.—Q. E. F.

PROP. IV. THEOREM.

GEN. ENUN.—If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal, *they shall likewise have their bases,* or third sides, *equal; and the two triangles shall be equal; and their other angles shall be equal, each to each; viz. those to which the equal sides are opposite.*

PART. ENUN.—Let ABC, DEF, be two △^{s}, which have the two sides AB, AC = to the two sides DE, DF, each to each; viz. AB = DE, and AC = DF, and also the ∠ BAC = ∠ EDF; *then* the base BC shall = base EF; and the area of the △ ABC shall = area of △ DEF; also the ∠ ABC shall = ∠ DEF; another ∠ ACB shall = ∠ DFE.



DEMONST.—1. For if the △ ABC be applied to the △ DEF, so that the pt. A may be on D, and the st. line AB upon DE; the pt. B shall coincide with the pt. E, because AB = DE. (Hyp.)

2. Also, AB coinciding with DE, AC shall fall upon DF, because the ∠ BAC = ∠ EDF. (Hyp.)

3. Also, pt. C shall coincide with pt. F, because AC = DF. (Hyp.)

4. Since therefore the pt. B was proved to

coincide with the pt. E, the base BC must coincide with the base EF; because, the pts. B and C coinciding respectively with E and F, if the base BC does not coincide with the base EF, but lies above or below it, two straight lines would inclose a space, which is impossible (Ax. 10); ∴ the base BC coincides with the base EF, and is = to it (Ax. 8).

5. ∴ the whole area of the $\triangle$ ABC coincides with the area of the $\triangle$ DEF, and is = to it: and the other $\angle^{s}$ of the one coincide with the remaining $\angle^{s}$ of the other, and are = to them; viz.:—

$\angle$ ABC = $\angle$ DEF; and $\angle$ ACB = $\angle$ DFE.

Therefore, if two $\triangle^{s}$ have two sides of the one = to two sides of the other, each to each, and have likewise the $\angle^{s}$ contained by those sides equal, their bases shall likewise be equal; and their areas shall be equal; and also their other $\angle^{s}$, to which the equal sides are opposite.—Q. E. D.

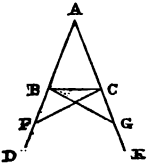
This Proposition involves the first condition of the perfect equality of two $\triangle^{s}$ in all their parts; and it extends of course to the equality of any number of $\triangle^{s}$ under the same circumstances. The method of proof adopted in it is not by *construction*, but by *supraposition: i. e.* by laying one $\triangle$ upon the other, and shewing that they coincide. It is again employed in the Eighth Proposition, which, together with the present, is the foundation of all that follows with respect to the comparison of $\triangle^{s}$. The student will be assisted in comprehending the Proposition and its proof by forming two = $\triangle^{s}$ of card-board, and placing one upon the other as directed in the text.

Upon this, and its cognate propositions, is based the Theory of *Perspective*.

PROP. V. THEOR.

Gen. Enun.—*The angles at the base of an* isosceles triangle *are equal;* and if the equal sides be produced, *the angles upon the other side of the base shall be equal.*

Part. Enun.—Let ABC be an isosc. △, of which the side AB = side AC (Def. 24); and let the st. lines AB, AC be produced to D and E (Post. 2); *then* ∠ ABC = ∠ ACB, and ∠ CBD = ∠ BCE.



Const.—In BD take any pt. F. From AE the > cut off AG = AF, the < (Prop. III)., and join FC, GB (Post. 1).

Demonst.—1. To prove the $\triangle^s$ AGB, AFC = in every respect.

Because AF = AG, and AB = AC; ∴ the two sides FA, AC = two sides GA, AB, each to each. Also they contain ∠ FAG, common to the two $\triangle^s$ AFC, AGB; ∴ the base FC = the base BG, and the area of △ AFC = area of △ AGB, and the remaining $\angle^s$ of the one = remaining $\angle^s$ of the other, to which the equal sides are opposite (Prop. IV.): *i. e.* ∠ ACF = ∠ ABG, and ∠ AFC = ∠ AGB.

2. To prove that BF = CG.

Again: because the whole AF = the whole AG (Const.), and the part AB = the part AC (Hyp.); ∴ the remainder BF = the remainder CG (Ax. 3).

3. To prove △ BFC = △ CGB.

Also it has been proved that FC = GB; ∴ in the $\triangle^{s}$ BFC, CGB, the two sides BF, FC = the two sides CG, GB, each to each, and ∠ BFC = ∠ CGB; ∴ △ BFC = △ CGB (Prop. IV.), and the remaining $\angle^{s}$ are =, each to each: *i. e.* ∠ FBC = ∠ GCB, and ∠ BCF = ∠ CBG.

4. To prove ∠ ABC = ∠ ACB.

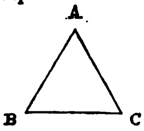
Now since it has been shown that the whole ∠ ABG = the whole ∠ ACF, and that the part ∠ CBG = the part ∠ BCF; ∴ the remaining ∠ ABC = remaining ∠ ACB (Ax. 3): *i. e.* the $\angle^{s}$ at the base of the △ ABC are = to one another.

5. And it has already been proved that the ∠ FBC = ∠ GCB, which are the $\angle^{s}$ on the other side of the base.

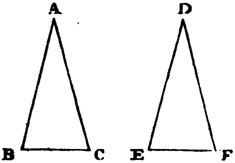
Therefore the $\angle^{s}$ at the base, &c.—Q. E. D.

COROLLARY.—Hence every equilateral △ is also equiangular.

The proof of this Corollary is very simple. Thus:—Let ABC be an equilateral △; then, because AB = AC, ∴ ∠ ABC = ∠ ACB; and because AB = BC, ∴ ∠ BAC = ∠ ACB; ∴ the $\angle^{s}$ ABC, ACB, BAC, are equal: *i.e.* the △ ABC is equiangular.—Q. E. D.



Partly from the figure, and partly from the difficulty which the demonstration presents to the beginner, the Fifth Proposition has obtained the ominous title of *Pons Asinorum*, or the *Asses' Bridge;* and it certainly requires some attention to surmount it. The former part of it may be very readily proved by the method of supraposition; and when Prop. XIII. has been proved, a still more simple demonstration of the whole will be given. In the meantime,

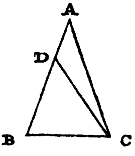


Let ABC, DEF be two isosceles △s, having the sides AB, AC of the one = to each other, and also = to the sides DE, DF of the other. Also let the included ∠ BAC = included ∠ EDF; *then* the ∠s B and C respectively = ∠s E and F. (Prop. IV.) But since AB = DF, and AC = DE, if the △ DEF be turned over, and applied to the △ ABC, so that the DF may coincide with AB, and DE with AC, the ∠ F will coincide with ∠ B, and the ∠ E with ∠ C, as in Prop. IV.; ∴ the ∠s B and C are each = E, and ∴ they = each other.—Q. E. D.

In forming roofs, brackets, tripods, and in various operations of Carpentry, this principle is constantly applicable.

PROP. VI. THEOR.

GEN. ENUN.—If two angles of a triangle be equal to one another, *the sides also which subtend,* or are opposite to, *the equal angles, shall be equal to one another.*



PART. ENUN.—Let ABC be a △, of which the ∠ ABC = ∠ ACB, *then* the side AB = AC.

CONST.—For if AB be not = AC, one must be > the other.

Suppose AB to be the >, and that from it is cut off DB = AC, the <. (Prop. III.) Join DC (Post. 1).

DEMONST.—1. Show the absurdity resulting from the above supposition; viz., that △ DBC would = △ ABC.

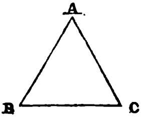
Since in $\triangle^s$ DBC, ACB, the side DB = AC, and BC is common to both, ∴ the two DB, BC = the two AC, CB, each to each; and the $\angle$ DBC = $\angle$ ACB (Hyp.); ∴ base DC = base AB, and the area of $\triangle$ DBC = area of $\triangle$ ACB: *i.e.* the $<\ =\ >$, which is absurd.

2. Hence AB is not unequal to AC: *i.e.* it is = to it.

Wherefore, if two $\angle^s$ of a triangle, &c.—Q. E. D.

Cor.—Hence every equiangular $\triangle$ is also equilateral.

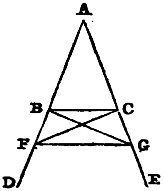
To prove this corollary, let the $\angle^s$ of the $\triangle$ ABC be all equal to one another: *then* because the $\angle$ ABC = $\angle$ ACB, ∴ AB = AC; and because the $\angle$ ABC = $\angle$ BAC, ∴ BC = AC; ∴ AB = AC = BC: *i. e.* the $\triangle$ ABC is equilateral.—Q. E. D.



The Sixth Proposition is proved by what is called a *reductio ad absurdum;* that is, by assuming an opposite conclusion to that which is enunciated, and showing it to be inconsistent with the hypothesis. It is the *converse* of the former part of the fifth ; and though in this case, as in others, Euclid has only demonstrated what was necessary to the proof of subsequent propositions, the second part is also convertible. It is not, however, to be supposed that, in taking the conclusion for an hypothesis, the original hypothesis becomes a necessary conclusion. Instances will be noticed in which this is not the case. In the meantime, the converse of the latter part of the Fifth Proposition may be thus demonstrated.

PROP. C. THEOR.

Gen. Enun.—If, when two sides of a triangle are produced, the angles below the base are equal to each other, *the two sides are also equal to one another.*



Part. Enun.—Let ABC be a △; and the sides AB, AC being produced to D and E, let the ∠ DBC = ∠ ECB; *then* the side AB shall = the side AC.

Const.—In BD take any pt. F, and from CE cut off CG = BF. (Prop. III.) Join BG, CF, FG.

Demonst.—1. Prove that the ∠ BCF = ∠ CBG.

Because BF = CG, and BC is common to the $\triangle^{s}$ FBC, GCB, ∴ the two sides FB, BC = GC, CB, each to each; and the contained ∠ FBC = ∠ GCB (Hyp.); ∴ the base FC = base GB, and the ∠ BCF = ∠ CBG. (Prop. IV.)

2. Prove that ∠ GCF = ∠ FBG.

But ∠ BCG = ∠ CBF (Hyp.); ∴ by subtraction, ∠ GCF = ∠ FBG (Ax. 5).

3. Prove that ∠ BFG = ∠ CGF.

Again, in $\triangle^{s}$ FBG, GCF, the sides FB, BG = GC, CF, each to each; and the ∠ FBG = ∠ GCF; ∴ ∠ BFG = ∠ CGF. (Prop. IV.)

4. Prove AF = AG.

Now these are the $\angle^{s}$ at the base of the △ AFG; ∴ AF = AG. (Prop. VI.)

5. Prove AB = AC.

But BF = CG (Const.); ∴ by subtraction, AB = AC. Wherefore, if, when two sides of a △ &c.—Q. E. D.

PROP. VII. THEOR.

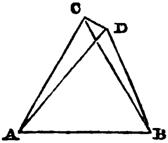
Gen. Enun.—Upon the same base, and on the same side of it, *there cannot be two triangles that have their sides, which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.*

PART. ENUN.—Let the two $\triangle^{s}$ ACB, ADB be upon the same base AB, and the same side of it, and let the sides CA, DA, terminated in A, be = ; *then* the sides CB, DB, terminated in B, cannot be =.

CONST.—Join CD.

DEMONST. (By *reduct. ad absurdum.*—If possible, suppose CB = DB.

Case 1. When the vertex of each △ is *without* the other △.



1. Prove that ∠ BDC is > ∠ BCD.

Because AC = AD (Hyp.), ∴ ∠ ACD = ∠ ADC. (Prop. V.) But ∠ ACD > ∠ BCD (Ax. 9); ∴ ∠ ADC > ∠ BCD. Much more is ∠ BDC > ∠ BCD.

2. Prove that, also, on the above supposition, ∠ BDC = ∠ BCD.

Now if BC = BD, ∴ ∠ BDC = ∠ BCD. (Prop. V.)

3. But it has been shown that ∠ BDC is > ∠ BCD, which is impossible.

Case 2. When one of the vertices, as D, is *within* the other △ ACB.

CONST.—Produce AC, AD to E and F, and join CD.

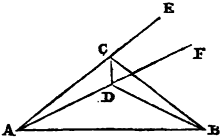
DEMONST. *(Ad abs.)*

1. Prove that ∠ BDC is > ∠ BCD.

Because AC = AD (Hyp.) in △ ACD, ∴

D

$\angle^{s}$ ECD, FDC, upon the other side of the base CD, are equal. (Prop. V.) But the ∠ ECD > ∠ BCD (Ax. 9); ∴ ∠ FDC > ∠ BCD. Much more is ∠ BDC > ∠ BCD.



2. Also that they are =.

Now, if in △ BCD, BC = BD, ∴ ∠ BDC = ∠ BCD. (Prop. V.)

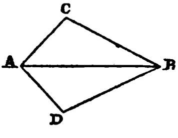
3. But it has been shown that ∠ BDC is > ∠ BCD, which is impossible.

The *third* case, in which the vertex of one △ is *upon* a side of the other, needs no demonstration. (See figure to Prop. VI.)

Wherefore, upon the same base, and on the same side of it, &c.—Q. E. D.

In the same way it may be proved that, if the sides BC, BD are equal, then AC, AD cannot be equal; and the student will do well to work out the demonstration. The proof includes what is called a demonstration *à fortiori*.

Euclid has established this theorem for the sole purpose of proving the next, to which it may therefore be considered as a *Lemma; i. e.* a proposition which is merely preparatory to the demonstration of another. The limitation that the two triangles should be upon the same side of the base is manifestly essential; for upon the other side of the base AB, a △ ABD may be constructed in every respect similar and = to the △ ABC. (Prop. IV.)



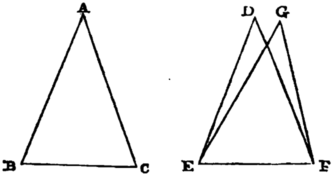
PROP. VIII. THEOR.

GEN. ENUN.—If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; *the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other.*

PART. ENUN.—Let ABC, DEF be two $\triangle^s$, having the sides AB, AC = to the two sides DE, DF, each to each; likewise let the base BC = base EF; *then* ∠ BAC shall be = ∠ EDF.

DEMONST.—(By supraposition).

1. For if the △ ABC be applied to the △ DEF, so that the pt. B be on E, and the st. line BC on EF, the pt. C shall coincide with F, because BC = EF. (Hyp.)



2. Now BC coinciding with EF, AB, AC shall also coincide with DE, DF.

For if AB, AC, do not coincide with DE, DF, but take another direction, as, for instance, GE, GF, then upon the same base EF, and upon the same side of it, there can be two $\triangle^s$ DEF, GEF, which have their sides terminated in one extremity of the base = to one another, and also those which are terminated in the other extremity, which is impossible. (Prop. VII.)

3. If ∴ the base BC coincides with EF, the sides AB, AC must coincide with DE, DF; and ∴ also the ∠ BAC coincides with ∠ EDF, and is = to it.

Wherefore, if two △s have two sides, &c.—Q. E. D.

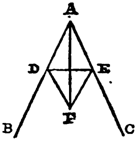
This Proposition is the converse of the former part of the fourth. It contains the second condition of equality of two or more △s; and the coincidence of the ∠s contained by the two equal sides being proved, the equality of the areas and the remaining ∠s follows from the Fourth Proposition.

PROP. IX. PROB.

Gen. Enun.—*To bisect* a given rectilineal angle; *that is, to divide it into two equal angles.*

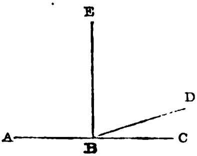
Part. Enun.—Let BAC be the gn. rectilineal ∠; *then* it is required to bisect it.

Const.—In AB take any pt. D, and from AC cut off AE = AD. (Prop. III.) Join DE; and upon it describe an equilateral △ DEF. (Prop. I.) Join AF. The st. line AF bisects the ∠ BAC.



Demonst.—Because AD = AE, and AF is common to △s DAF, EAF, ∴ the two sides DA, AF = EA, AF, each to each; and the base DF = base EF (Const.); ∴ ∠ DAF = ∠ EAF. (Prop. VIII.)

line, the ∠ DBE = ∠ EBA (Def. 10); ∴ the ∠ DBE = ∠ CBE: *i. e.* the < = >, which is impossible. Therefore two straight lines cannot have a common segment.—Q. E. D.



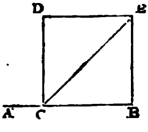
This corollary is placed by Euclid himself among the Axioms, and the simple proof, deduced from the Eleventh Proposition, was added by Dr. Simson. From the plain fact, however, that two straight lines cannot meet in more than one pt. without coinciding altogether, it is clear that a *segment*, or part cut (*seco*) from one straight line, cannot possibly belong to another also. With respect to the Proposition itself, it is clear that, if the gn. pt. be at or near the extremity of the line, it will be necessary to produce it: but if such production be impossible or inconvenient, the following method may be used:—

PROP. D. PROB.

GEN. ENUN.—Through the extremity of a given *finite* straight line *to draw a straight line at rt.* $\angle^s$ *to it.*

PART. ENUN.—Let AB be a gn. finite st. line; *then* it is required to draw through B a st. line at rt. $\angle^s$ to AB.

CONST.—In AB take any pt. C, and through C draw CD at rt. $\angle^s$ to AB (Prop. XI.), and make CD = CB. (Prop. III.) Bisect the ∠ BCD in the st. line CE. (Prop. IX.) Again, through D draw DE at rt. $\angle^s$ to CD (Prop. XI.), meeting CE in E. Join BE (Post. 1). *Then* BE will be at rt. $\angle^s$ to AB.



DEMONST.—Because CB = CD, and CE is common to the $\triangle^s$ ECB, ECD; ∴ the two sides BC, CE = DC, CE, each to each, and the ∠ BCE = ∠ DCE; ∴ the ∠ CBE = ∠

CDE = rt. ∠. (by Const.) Wherefore through B, its extremity, a line BE, has been drawn at rt. ∠s to AB.—Q. E. F.

From the Eleventh Proposition, the two following are also deductions.

PROP. E. PROB.

Gen. Enun.—In a straight line given in position, but indefinite in length, *to find a point equally distant* from two given points, either on the same or on opposite sides of the given line.

Fig. 1.

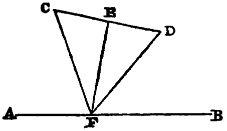
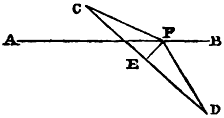


Fig. 2.



Part. Enun.—Let AB be the gn. line, and C, D the gn. pts.; *then* it is required to find in the st. line AB a pt. equidistant from C and D.

Const.—Join CD, and bisect it in E (Prop. X.); through E draw EF at rt. ∠s to CD. (Prop. XI.) F is the point required. Join CF, DF.

Demonst.—Because CE = ED, and EF is common to △ CEF, DEF; ∴ the two sides CE, EF = DE, EF, each to each, and the ∠ CEF = ∠ DEF (Def. 10); ∴ the base CF = base DF. (Prop. IV.)

Wherefore a pt. F has been found in the gn. line AB, equidistant from the gn. pts. C and D.—Q. E. F.

PROP. F. THEOR.

Gen. Enun.—If the three sides of a triangle be bisected, and straight lines be drawn through the points of bisection at rt. angles to the sides, *these lines will meet in a point equally distant from each of the angles.*

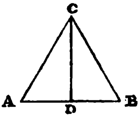
Wherefore the ∠ BAC is bisected by the st. line AF.—Q. E. F.

It is clear that the same result would follow from describing an isosceles △ upon DE (Prop. A or B); and that by repeating the operation, a rectilineal ∠ may be divided in 4, 8, 16, 32, 64 equal parts; and so on, doubling continually. A similar remark applies to the following Proposition.

PROP. X. PROB.

GEN. ENUN.—*To bisect* a given finite straight line; *that is, to divide it into two equal parts.*

PART. ENUN.—Let AB be the gn. st. line; *then* it is required to bisect it.



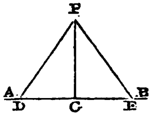
CONST.—Describe upon AB an equilateral △ ABC (Prop. I.), and bisect the ∠ ACB by the st. line CD (Prop. IX.); *then* AB is cut into two = parts in the pt. D.

DEMONST.—Because AC = CB, and CD is common to the △ˢ ACD, BCD, ∴ the two sides AC, CD = BC, CD, each to each, and the ∠ ACD = ∠ BCD (Const.); ∴ the base AD = base DB (Prop. IV.): *i. e.* the st. line AB is bisected in the point D.—Q. E. F.

PROP. XI. PROB.

GEN. ENUN.—*To draw a straight line at right angles* to a given straight line from a given point in the same.

PART. ENUN.—Let AB be the gn. st. line, and C a gn. pt. in it; *then* it is required to draw from C a st. line at rt. $\angle^s$ to AB.



CONST.—In AC take any pt. D, and make CE = CD. (Prop. III.) Upon DE describe an equilateral △ DEF. (Prop. I.) Join CF (Post. 1). The st. line CF, drawn through the gn. pt. C, is at rt. $\angle^s$ to AB.

DEMONST.—1. Because DC = CE, and CF is common to the two $\triangle^s$ DCF, ECF; ∴ the two sides DC, CF = EC, CF, each to each; also base DF = base EF; ∴ ∠ DCF = ∠ ECF (Prop. VIII.), and they are adjacent $\angle^s$.

2. But when the adjacent $\angle^s$ which one st. line makes with another are equal, each of them is called a rt. ∠ (Def. 10); ∴ each of the $\angle^s$ DCF, ECF is a rt. ∠.

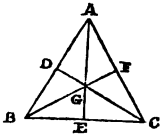
3. Wherefore from the gn. pt. C, in the gn. st. line AB, the line CF has been drawn at rt. $\angle^s$ to AB.—Q. E. F.

COR.—By help of this Proposition, it may be demonstrated that two straight lines *cannot have a common segment.*

Suppose, if possible, the two st. lines ABC, ABD to have the segment AB common to both of them.

From the pt. B draw BE at rt. $\angle^s$ to AB (Prop. XI.); *then,* because ABC is a st. line, the ∠ CBE = ∠ EBA; and if ABD is a st.

Part. Enun.—Let the sides of any △ ABC be bisected in the pts. DEF; through D and E let DG, EG be drawn at rt. $\angle^s$ to AB, BC, and meeting in G (Prop. XI.); *then* the st. line drawn through F at rt. $\angle^s$ to AC shall also pass through G, and this pt. G is equidistant from each of the $\angle^s$ of the △ ABC.



Const.—Join GF, GA, GB, GC (Post. 1).

Demonst.—1. Prove that AG = BG.

Because AD = DB, and DG is common to the $\triangle^s$ ADG, BDG; ∴ the two sides AD, DG = BD, DG, each to each, and the ∠ ADG = ∠ BDG (Def. 10); ∴ base AG = base BG. (Prop. IV.)

2. In the same manner it may be shewn that CG = BG.

3. ∴ also AG = CG.

4. Prove that ∠ AFG = ∠ CFG = rt. ∠.

Again, because AF = FC, and FG is common to the $\triangle^s$ AFG, CFG; ∴ the two sides AF, FG = CF, FG, each to each, and the base AG = base CG; ∴ also ∠ AFG = ∠ CFG (Prop. VIII.), and they are adjacant $\angle^s$; ∴ each of them is a rt. ∠.

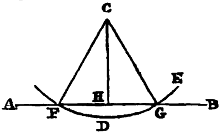
5. Hence the line FG, drawn at rt. ∠ to AC through F, passes through G, and it has been shown that AG = BG = CG.

Wherefore, if the three sides of a △ be bisected, &c. —Q. E. D.

PROP. XII. PROB.

Gen. Enun.—*To draw a straight line perpendicular to* a given straight line *of an unlimited length,* from a given point without it.

Part. Enun.—Let AB be the gn. st. line, which may be produced to any length both ways, and let C be a gn. pt. without it; *then* it is required to draw a st. line ⊥ to AB from the pt. C.



Const.—Take any pt. D upon the other side of AB. From cr. C, with rad. CD, describe the ⊙ EGF, meeting AB in the pts. F and G (Post. 3). Bisect FG in H (Prop. X.), and join CH. The line CH, drawn from the pt. C, is ⊥ to the gn. st. line AB. Join CF, CG.

Demonst.—Because FH = HG, and HC is common to the $\triangle^s$ FHC, GHC; ∴ the two sides FH, HC = GH, HC, each to each; and by the property of ⊙, the base FC = the base CG (Def. 15); ∴ the ∠ CHF = ∠ CHG (Prop. VIII.), and they are adjacent $\angle^s$: but when a st. line standing on another st. line makes the adjacent ∠ = to one another, each of them is a rt. ∠, and the st. line which stands upon the other is called a ⊥ to it; ∴ from the gn. pt. C a st. line CH has been drawn $\perp^r$ to AB. —Q. E. F.

The condition that the st. line should be given in position only, and not in length, is manifestly essential; for otherwise the gn. pt. might be so placed, not only that a ⊙ could not be described cutting the line in two pts., but the ⊥ would fall beyond the extremity of the line. In order to drop a ⊥ upon a gn. *finite* line at or near its extremity, the 31st Prop. of Book III. must first be proved.

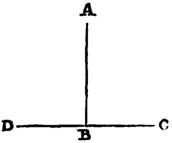
PROP. XIII. THEOR.

GEN. ENUN.—The angles, which one straight line makes with another upon one side of it, *are either two right angles, or are together equal to two right angles.*

PART. ENUN.—Let the st. line AB make with CD, upon one side of it, the $\angle^s$ CBA, ABD; *then* these are either two rt. $\angle^s$, or their sum = two rt. $\angle^s$.

If the ∠ CBA = ∠ ABD, as in *fig.* 1, *then* each of them is a rt. ∠ (Def. 10). But, if not,

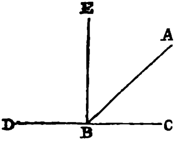
Fig. 1.



CONST.—From the pt. B (*fig.* 2) draw BE at rt. $\angle^s$ to CD (Prop. XI.), so that the ∠ CBE, EBD are two rt. $\angle^s$.

DEMONST.—1. Now since the ∠ CBE = ∠ CBA + ABE, add to each the ∠ EBD; ∴ ∠ CBE + EBD = ∠ CBA + ABE + EBD (Ax. 2).

Fig. 2.



2. Again, since the ∠ DBA = ∠ DBE + EBA, add to each the ∠ ABC; ∴ ∠ DBA + ABC = ∠ DBE + EBA + ABC (Ax. 2).

3. ∴ ∠ DBA + ABC = CBE + EBD (Ax. 1) = two rt. $\angle^s$.

Wherefore the $\angle^s$ which one st. line, &c.—Q. E. D.

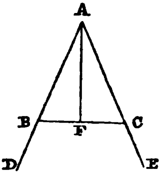
It is clear from this Proposition that the sum of all the

$\angle^{s}$, which are made by any number of st. lines drawn to the *same* pt. in another st. line, on the same side of it, = two rt. $\angle^{s}$.

The Proposition is of great practical use in *Trigonometry* and *Astronomy*.

We are now in a position to furnish a very simple demonstration of the equality of the $\angle^{s}$ above and below the base of an isosc. △. Thus:—

PART. ENUN.—Let ABC be an isosc. △, having the side AB = the side AC; and let the sides AB, AC be produced to D and E; *then* the ∠ ABC = ∠ ACB, and the ∠ DBC = ∠ ECB.



CONST.—Bisect the ∠ BAC by the st. line AF.

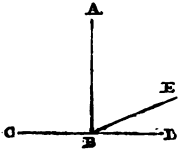
DEMONST.—1. Because AB = AC, AF is common to $\triangle^{s}$ ABF, ACF, and ∠ BAF = ∠ CAF; ∴ the $\triangle^{s}$ are =, and the ∠ ABC = ∠ ACB: *i. e.* the $\angle^{s}$ above the base are =.

2. Again, the $\angle^{s}$ CBA + CBD = two rt. $\angle^{s}$ = $\angle^{s}$ BCA + BCE (Prop.XIII.); but ∠ ABC = ∠ ACB; ∴ remaining ∠ DBC = remaining ∠ ECB: *i. e.* $\angle^{s}$ below the base are =.—Q. E. D.

PROP. XIV. THEOR.

GEN. ENUN.—If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, *these two straight lines shall be in one and the same straight line.*

PART. ENUN.—At the pt. B in the st. line AB, let the two st. lines BC, BD make the adjacent ∠ ABC, ABD = two rt. $\angle^{s}$; *then* BC and BD are in the same st. line.



DEMONST. (*Ad absurd.*) :—

For, suppose that BD be not in the same st. line with BC, and that BE is in the same st. line with it.

1. Then, because AB makes with CBE, upon one side of it, the $\angle^s$ CBA, ABE ; ∴ the $\angle^s$ CBA + ABE = two rt. $\angle^s$. (Prop. XIII.)

2. But the $\angle^s$ CBA + ABD = two rt. $\angle^s$. (Hyp.)

3. ∴ $\angle^s$ CBA + ABE = $\angle^s$ CBA + ABD (Ax. 1).

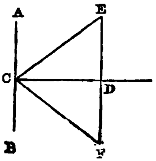
4. ∴ (by subtracting the common ∠ CBA) the ∠ ABE = ∠ ABD (Ax. 3) : *i. e.* the < = >, which is impossible.

5. ∴ BE is not in the same st. line with BC.

6. In like manner it may be proved that no other can be in the same st. line with it but BD, which is ∴ in the same st. line with it.

Wherefore, if at a pt. &c.—Q. E. D.

This Proposition is the *converse* of the preceding. It is necessary that the two lines BC, BD should meet *from opposite sides* of AB; for it is very possible that on the *same side* two lines may form $\angle^s$ = to two rt. $\angle^s$ with it, without being in the same st. line. Thus :—



CONST.—Let C be any pt. in a st. line AB. Draw CD. at rt. $\angle^s$ to AB. (Prop. XI.) Bisect the ∠ ACD by the st. line CE. (Prop. IX.) In CD take any pt. D, and through D draw DE at rt. $\angle^s$ to CD. Produce ED to F, and make DF = ED. (Prop. III.) Join CF.

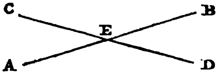
DEMONST.—*Then* in Δ^s CDE, CDF, ED, DC = FD, DC, each to each, and the rt. ∠ EDC = rt. ∠ FDC; ∴ base CE = base CF (Prop. IV.); and the ∠ ECD = ∠ FCD = half a rt. ∠ (Const.): ∴ the ∠ ACF = a rt. ∠ + half a rt. ∠, and the ∠ ACF + ACE

= two rt. $\angle^s$; and they are formed by two st. lines EC, FC, meeting AB in the same pt. The said lines, however, are not in the same st. line.

PROP. XV. THEOR.

Gen. Enun.—If two straight lines cut one another, *the vertical, or* opposite, *angles shall be equal.*

Part. Enun.—Let the two st. lines AB, CD cut one another in the pt. E; *then* the ∠ AEC shall be = ∠ DEB, and the ∠ CEB shall be = ∠ AED.



Demonst.—1. Because the st. line AE makes, with CD, the $\angle^s$ CEA, AED; ∴ these $\angle^s$ are = two rt. $\angle^s$. (Prop. XIII.)

2. Also, because the st. line DE makes, with AB, the $\angle^s$ AED, DEB; ∴ these $\angle^s$ are also = two rt. $\angle^s$. (Prop. XIII.)

3. ∴ the $\angle^s$ CEA + AED = $\angle^s$ AED + DEB (Ax. 1).

4. ∴ (by subtraction) the ∠ CEA = ∠ DEB.

5. In like manner it may be shown that ∠ CEB = ∠ AED.

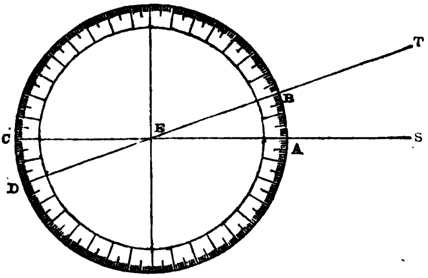
Wherefore, if two st. lines, &c.—Q. E. D.

Cor. 1.—From this it is manifest that, if two straight lines cut one another, the $\angle^s$ which they make at the pt. of intersection are together = to four rt. $\angle^s$.

Cor. 2.—Consequently, the sum of all the $\angle^s$ made by any number of st. lines intersecting at one pt. is = to four rt. $\angle^s$.

As the student should take nothing for granted, he will do well to demonstrate the equality of the $\angle^s$ CEB and AED, as stated in the Proposition.—Though capable of a rigorous demonstration, the Proposition is nevertheless but little more than an inference from the definition of an $\angle$. For if two straight lines, inclined at an $\angle$, intersect, their inclination must continue the same on both sides; or, in other words, the vertical $\angle^s$ will be $=$. With respect to the Corollaries, it is clear that CE makes, with AB, the $\angle^s$ AEC + CEB = two rt. $\angle^s$, and DE makes, with AB, the $\angle^s$ AED + DEB = two rt. $\angle^s$ (Prop. XIII.); ∴ by addition, the $\angle^s$ AEC + CEB + AED + DEB = four rt. $\angle^s$. Again, as these four $\angle^s$ are merely subdivided by any number of st. lines intersecting in E, the sum of four rt. $\angle^s$ remains unaltered. The practical application of the 15th Proposition to the measurement of $\angle^s$ may be thus illustrated:—

Let ABCD be a graduated ⊙ (Obs. on Def. 18), having an object-glass, or telescope, moveable round the cr. E; let the glass be directed to the object S, and the degree marked to which it points on the $\bigcirc^{ce}$; *then* let it be turned round till it points to another object T, and the degree marked as before. The $\angle^r$ distance between the objects, or the number of degrees which the glass has passed over, is clearly that which is contained in the $\angle$ AEB, measured by the arc AB; or, which is the same thing, the number contained in the $\angle$ CED, measured by the arc CD. Upon this principle the *Theodolite*, and several astronomical instruments are constructed.

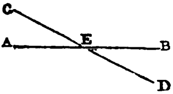


The converse of the Proposition may be thus demonstrated:—

PROP. G. THEOR.

Gen. Enun.—If at a point in a straight line, two other straight lines, upon the opposite sides of it, make the vertical angles equal, *these two straight lines shall be in one and the same straight line.*

Part. Enun.—At the pt. E, in the st. line AB, let the two st. lines CE, DE make the vertical $\angle^{s}$ CEA, DEB = to one another; *then* CE shall be in the same st. line with DE.



Demonst.—1. Because the ∠ CEA = = DEB (Hyp.), add to each the ∠ CEB; ∴ $\angle^{s}$ CEA + CEB = $\angle^{s}$ CEB and DEB (Ax. 2).

2. But the ∠ CEA + CEB = two rt. $\angle^{s}$ (Prop. XIII.); ∴ the ∠ CEB + DEB = two rt. $\angle^{s}$ (Ax. 1).

3. ∴ CE is in the same st. line with DE. (Prop. XIV.)

4. In the same manner, if the ∠ CEB = ∠ AED, it may be shown that CE and DE are in the same st. line.

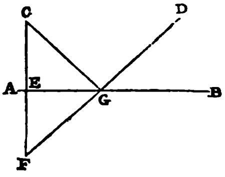
Wherefore, if at a pt. &c.—Q. E. D.

As a deduction from the 15th Proposition may be proved—

PROP. H. PROB.

Gen. Enun.—From two given points on the same side of a straight line given in position, *to draw two straight lines which shall meet the given line in the same point, and make equal angles with it.*

Part. Enun.—Let AB be the gn. st. line, and C, D two gn. pts. on the same side of it; *then* it is required, &c.



Const.—From C draw CE ⊥ to AB (Prop. XII.); produce CE to F (Post. 2), and make EF = CE (Prop. III.); join DF, CG (Post. 1); *then* CG, DG will be the lines required.

Demonst.—Because CE = FE, and EG is common to the $\triangle^{s}$ CEG, FEG; ∴ the two sides CE, EG = FE, EG, each to each, and the rt. ∠ CEG = rt. ∠ FEG; ∴ the ∠ CGE = ∠ FGE (Prop. IV.) = vertical ∠ DGB (Prop. XV.): *i. e.* the two st. lines CG, DG, drawn from C, D, and meeting in G, make = $\angle^{s}$ with AB.—Q. E. F.

PROP. XVI. THEOR.

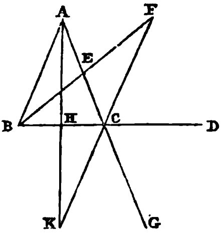
Gen. Enun.—If one side of a triangle be produced, *the exterior angle is greater than either of the interior opposite angles.*

Part. Enun.—Let a side BC of any △ ABC be produced to D; *then* the exterior ∠ ACD is > than either of the exterior and opposite $\angle^{s}$ CBA, BAC.

Const.—Bisect AC in E (Prop. X.); join BE, and produce it to F, making EF = BE (Prop. III.); and join FC.

Demonst.—1. Prove that in the $\triangle^{s}$ EAB, ECF, the ∠ BAE = ∠ ECF.

Because AE = EC, and BE = EF (Const.); ∴ the two sides AE, EB = CE, EF, each to each, and the vertical ∠ AEB = ∠ CEF (Prop. XV.); ∴ the base AB = base CF, and the △ AEB = △ CEF, and the remaining $\angle^{s}$ of the one = the remaining $\angle^{s}$ of the other, to which the equal sides are opposite; ∴ the ∠ BAE = ∠ ECF. (Prop. IV.)



2. But the ∠ ACD is > ∠ ECF; ∴ the ∠ ACD is > ∠ BAC.

3. In the same manner it may be shown, by producing AC to G, and bisecting BC, that the ∠ BCG, which is equal to the vertically opposite ∠ ACD (Prop. XV.), is > than the ∠ ABC.

Wherefore, if one side of a △ be produced, &c.—Q. E. D.

The second part of the Proposition is proved thus:—

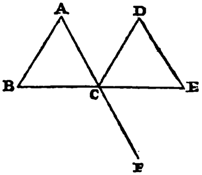
CONST.—Produce AC to G; bisect BC in H; join AH, and produce it to K, making HK = AH; and join KC.

DEMONST.—1. Then, because BH = HC, and AH = HK in the △s AHB, CHK; ∴ the two sides BH, HA = CH, HK, each to each, and the vertical ∠ BHA = ∠ CHK (Prop. XV.); ∴ the ∠ ABH = ∠ HCK.

2. But the ∠ BCG is > ∠ HCK; ∴ the ∠ BCG is > ABC.

3. ∴ also the vertical ∠ ACD is > ABC.—Q. E. D.

It is observable that the exterior ∠ ACD will be either >, =, or < than the adjacent ∠ ACB, according as the △ ABC is acute-angled, right-angled, or obtuse-angled respectively.



The following is a practical illustration of the property set forth in the Proposition:—

Let the △ ABC be moved along the line BE till B coincides with C. The vertex of the △ will now be to the right of A, so that DC will be *within* the exterior ∠ ACE. In other words, the interior ∠ ABC will have moved into the position of the ∠ DCE, which is < the ∠ ACE.

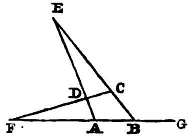
In like manner, if the △ be moved along the line AF, the ∠ A will be seen to be less than the ∠ BCF, which = ∠ ACE.

This Proposition is one of those which are not convertible; for it does not follow that, if the exterior ∠ of a rectilineal figure be > than its interior opposite ∠s, the figure is necessary a △. This will be evident from the following Theorem:—

PROP. K. THEOR.

GEN. ENUN.—If a trapezium, having two of its adjoining angles an obtuse and a right angle respectively, be such that its opposite sides, produced in the direction of these angles, shall meet, *the exterior angle in the other direction shall be greater than any of the three interior opposite angles.*

PART. ENUN.—Let ABCD be a trapezium, of which the ∠ BAD is obtuse, and the ∠ ADC a rt. ∠ ; and let the sides AD, BC be produced to meet in E, and BA, CD in F; *then*, if AB be produced towards G, the exterior ∠ CBG shall be > than either of the three ∠s BAD, ADC, DCB.



DEMONST.—1. For the exterior ∠ EBG of △ ABE, is > interior ∠ EAB. (Prop. XVI.)

2. But ∠ EAB is an obtuse ∠ ; ∴, *a fortiori*, ∠ CBG is > the rt. ∠ ADC (Def. 11).

3. Again, in the △ CFB, the exterior ∠ CBG is > interior ∠ FCB. (Prop. XVI.)

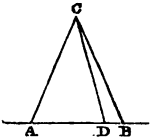
4. ∴ the exterior ∠ CBG is > either of the interior ∠s BAD, ADC, DCB.

Wherefore, if a trapezium, &c.—Q. E. D.

The student cannot be too frequently reminded that the Propositions of Euclid are of universal application, and in this 16th Proposition he will find it a useful exercise to prove its truth with respect to the other exterior ∠s, formed by producing CB towards B, and the sides AB and AC both ways. By way of deduction may be proved,

PROP. L. THEOR.

GEN. ENUN.—*Only two equal straight lines can be drawn* to another straight line from a given point without it.



Part. Enun.—Let AB be a st. line; C a gn. pt. without it; and CA, CB two = st. lines drawn from C to AB; *then* no other st. line can be drawn from C to AB, which shall be = to CA or CB.

Demonst. (*Ad abs.*)

If possible, suppose CD = CA = CB.

1. Now, since CA = CB, ∴ ∠ CAB = ∠ CBA; and because CA = CD, ∴ ∠ CAB = ∠ CDA (Prop. V.); ∴ ∠ CDA = ∠ CBA (Ax. 1).

2. But in the △ CDB, exterior ∠ CDA is > interior ∠ CBA (Prop. XVI.), being also = to it, which is absurd.

3. ∴ CD is not = to CA or CB.

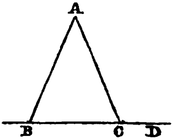
4. In the same manner it may be shown that no other line drawn from the gn. pt. C to AB is = to them.

Wherefore, only two &c.—Q. E. D.

Cor.—A circle cannot cut a st. line in more pts. than two.

PROP. XVII. THEOR.

Gen. Enun.—*Any two angles of a triangle are together less* than two right angles.



Part. Enun.—Let ABC be any △. *Then* any two of its ∠ˢ are together less than two rt. ∠ˢ.

Demonst.—Produce BC to D (Post. 2).

1. Now the exterior ∠ ACD of the △ ABC is > interior opposite ∠ ABC. (Prop. XVI.) To each of these add the ∠ ACB; ∴ the ∠ˢ ACD + ACB are > the ∠ˢ ABC + ACB (Ax. 4).

2. But the ∠ˢ ACD + ACB = two rt. ∠ˢ

(Prop. XIII.); ∴ the $\angle^s$ ABC + ACB are < two rt. $\angle^s$.

3. In like manner it may be proved that the $\angle^s$ BAC + ACB, and the $\angle^s$ CAB + ABC, are < two rt. $\angle^s$.

Therefore any two angles &c.—Q. E. D.

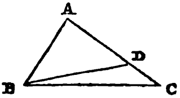
This and the preceding Proposition are carried further in Prop. XXXII. It will be well to prove it in all its cases by producing each side successively.

The following are Corollaries:—That the $\angle^s$ at the base of an isosc. △ are acute; and that, if one ∠ of a △ be right or obtuse, the two others must be acute.

PROP. XVIII. THEOR.

Gen. Enun.—The greater side of every triangle is *opposite to the greater angle.*

Part. Enun.—Let ABC be any △ of which the side AC is > than AB; *then* ∠ ABC shall be > the ∠ ACB.



Const.—Because AC is > AB, cut off AD = AB. (Prop. III.) Join BD.

Demonst.—1. Then the exterior ∠ ADB of the △ DBC is > interior ∠ DCB. (Prop. XVI.)

2. Now, because AB = AD, ∴ ∠ ABD = ∠ ADB (Prop. V.); ∴ ∠ ABD is > ∠ ACB.

3. ∴, *a fortiori,* ∠ ABC is > the ∠ ACB.

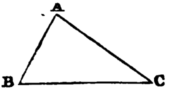
Wherefore the greater side &c.—Q. E. D.

It appears from this Proposition that all the $\angle^s$ of a scalene △ are unequal. If BC be also greater than AB, it may be proved, in like manner, that the ∠ BAC is greater than the ∠ ACB; and that as BC is > or < AC, the ∠ BAC is > or < the ∠ ABC.

PROP. XIX. THEOR.

Gen. Enun.—The greater angle of every triangle *is subtended by the greater side, or has the greater side opposite to it.*

Part. Enun.—Let ABC be a △, of which the ∠ ABC is > the ∠ ACB; *then* the side AC shall be > AB.



Demonst.—For, if not >, AC must either be = AB, or < than it.

1. It is not =, for then the ∠ ABC would be = ∠ ACB (Prop. V.), but it is not.

2. Neither is it <, for then the ∠ ABC would be < the ∠ ACB (Prop. XVIII.), but it is not.

3. ∴ the side AC is neither = nor < AB; *i.e.* AC is > AB.

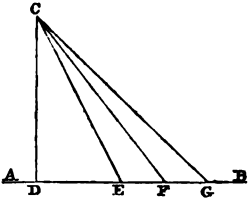
Wherefore the greater angle &c.—Q. E. D.

In like manner, the relative lengths of the other sides may be proved; and hence may be deduced the following:—

PROP. M. THEOR.

Gen. Enun.—Of all straight lines which can be drawn from a given point to a straight line of unlimited length, a perpendicular *is the shortest*, and that which is nearer to the perpendicular *is shorter than one more remote.*

Part. Enun. and Const.—Let AB be a st. line of unlimited length, C the gn. pt. From C draw CD ⊥ to AB (Prop. XII.), and draw CE, CF, CG, &c. to any other pts. along the line AB; *then* CD is the shortest line which can be drawn from C to AB, CE shorter than CF, CF than CG, and so on.



Demonst.—1. Because ∠ CDE is a rt. ∠, the exterior ∠ CEF is > a rt. ∠ (Prop. XVI.); ∴ CED is < a rt. ∠ (Prop. XIII.), < ∠ CDE; ∴ CD is < CE. (Prop. XIX.)

2. In the same manner it may be shown that CD is < every other line drawn from C to AB.

3. Again, because the ∠ CEF is an obtuse ∠; ∴ the ∠ CFE is an acute ∠; ∴ CE is less CF. (Prop. XIX.)

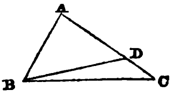
4. In the same manner, CF is < CG, &c., &c.

Therefore, of all st. lines &c.—Q. E. D.

PROP. N. THEOR.

Gen. Enun.—Any side of a triangle *is greater than the difference between the other two sides.*

Part. Enun.—Let ABC be a △. *Then* any of its sides is greater than the difference between the other two sides.



Const.—Let AC be > AB, and make AD = AB. (Prop. III.) Join BD (Post. 1).

Demonst.—1. Because AB = AD, ∴ the ∠ ABD = ∠ ADB. (Prop. V.)

2. Now the exterior ∠ BDC of △ ABD is > the interior ∠ ABD (Prop. XVI.), *i. e.* > ∠ ADB; ∴, *a fortiori*, > ∠ DBC (Prop. XVI.); ∴ the side BC is > the side DC (Prop. XIX.), > AC − AD, > AC − AB.

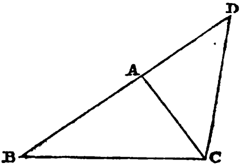
3. In the same way it may be shown that AB is > AC − BC, and AC > BC − AB.

Wherefore any side &c.—Q. E. D.

PROP. XX. THEOR.

Gen. Enun.—Any two sides of a triangle are together *greater than the third side.*

Part. Enun.—Let ABC be any △. *Then* the sum of any two of its sides is > the third side, viz., BA + AC is > BC, AB + BC > AC, and BC + CA > AB.



Const.—Produce BA to the pt. D (Post. 2), making AD = AC. (Prop. III.) Join DC. Then,

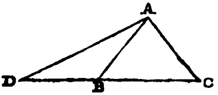
Demonst.—1. Because DA = AC, ∴ ∠ ADC = ∠ ACD (Prop. V.); but the ∠ BCD is > the ∠ ACD (Ax. 9); ∴ the ∠ BCD is > the ∠ ADC; and the side BD is > the side BC. (Prop. XIX.)

2. But BD = BA + AD = BA + AC (Const.); ∴ BA + AC is > than BC.

3. In the same manner, it may be proved that AB + BC is > AC, and that BC + CA is > AB.

Wherefore, any two sides &c.—Q. E. D.

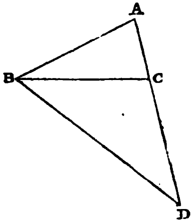
This is one of those Propositions of which it has been said that the proof is unnecessary, since any person would readily admit its truth without any demonstration. This may be very true: but the proof is still essential to the completeness of a connected mathematical system; and the student must be able to work the two remaining cases.



In order, therefore, to prove that AB + BC are > AC, produce CB to D, make BD = AB, and join AD. *Then* because BA = BD, ∴ ∠ BDA = ∠ BAD (Prop. V.): but the ∠ DAC is > the ∠ BAD (Ax. 9); ∴

the ∠ DAC is > ∠ CDA; ∴ in the Δ ADC, the side DC is > side AC (Prop. XIX.) Now DC = DB + BC = AB + BC (Const.); ∴ AB + BC is > AC.

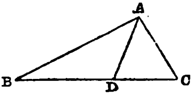
Again, to prove that BC + CA are > AB, produce AC to D, make CD = BC, and join BD. *Then*, because CD = BC, ∴ ∠ CBD = ∠ CDB (Prop. V.): but the ∠ ABD is > ∠ CBD (Ax. 9); ∴ the ∠ ABD is > ∠ ADB; ∴ AD is > AB (Prop. XIX.): but AD = AC + CD = AC + CB; ∴ AC + CB is > AB.



Another solution of the Proposition, without producing the side, may not be unacceptable. Thus:—

Let ABC be a Δ. Any two sides, as BA + AC, are > BC.

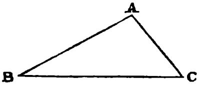
Bisect the ∠ BAC by the st. line AD (Prop. IX.); *then* the exterior ∠ ADB of the Δ ADC is > the interior opposite ∠ DAC (Prop. XVI.); ∴ also the ∠ ADB is > the ∠ DAB; ∴ in the Δ ADB, the side AB is > BD.



In like manner it may be shown that AC is > DC; ∴ AB + AC are > BD + DC, *i. e.* > BC.

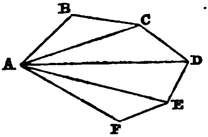
Of this Proposition the following are Corollaries:—

COR. 1.—The three sides of a Δ, taken together, *exceed the double of any one side*, but are *less than the double of any two sides*. For the sides AB + AC are > BC; add BC to each; ∴ AB + AC + BC are > twice BC. But the side AB is < AC + BC; add AC + BC to each; ∴ AB + AC + BC is < twice AC + twice BC, or 2 (AC + BC).



COR. 2.—Any one side of a rectilineal figure is *less than the sum of the remaining sides.*

Let ABCDEF be any rectilineal figure; divide it into Δ^s by joining AC, AD, AE, and so on for any number

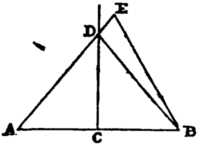


of sides; *then* AF is < AE + EF, AE < AD + DE, AD < AC + CD, and AC < AB + BC; ∴, *a fortiori*, AF is < AB + BC + CD + DE + EF.

The following deductions belong here:—

PROP. O. THEOR.

Gen. Enun.—If a perpendicular be drawn bisecting a given straight line, any point in this perpendicular *is at equal distances*, and any point without the perpendicular *is at unequal distances from the extremities of the line.*



Part. Enun.—Let AB be a given st. line. Bisect it* in C, and through C draw CD at rt. $\angle^{s}$ to it. *Then* any pt. D upon the ⊥ CD is $=^{ly}$ distant from A and B; and any pt. E, without the ⊥, is not $=^{ly}$ distant from A and B.

Const.—Join AD, BD.

Demonst.—1. Because AC = CB, and CD is common to $\triangle^{s}$ ACD, BCD; ∴ the sides AC, CD = BC, CD, each to each, and the rt. ∠ ACD = rt. ∠ BCD; ∴ the base AD = the base BD (Prop. IV.): and the same may be proved of the lines drawn from any other pt. in CD to A and B.

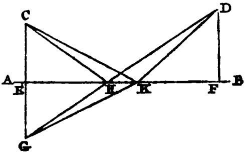
2. Again, if E be a pt. without the ⊥, join EA, EB, DB. *Then*, as before, AD = DB; ∴ AE = AD + DE = BD + DE. But BD + DE are > BE (Prop. XX.); ∴ AE is > BE: and the same may be proved of any other pt. on either side of CD.

Wherefore, if a ⊥ be drawn &c.—Q. E. D.

PROP. P. PROB.

Gen. Enun.—From two given points on the same side of a line given in position, *to draw two lines which shall meet in a point in this line, so that their sum shall be less* *than that of any other lines drawn* from the two given points *to any other point in the same line.*

PART. ENUN.—Let AB be a line gn. in position, and C, D two gn. pts. on the same side of it. *Then* it is required to draw from the pts. C, D, two st. lines which shall meet in a pt. in AB, and of which the sum shall be less than that of any other two lines drawn from C and D to any other pt. in AB.



CONST.—From C and D draw CE, DF, ⊥ to AB (Prop. XII.); produce CE to G, making EG = CE (Prop. III.); join DG, cutting AB in H, and join CH. *Then* CH, DH are the lines required.

In AB take any other pt. K, and join CK, DK, GK.

DEMONST.—1. Now in the $\triangle^{s}$ CEH, GEH, the two sides CE, EH = GE, EH, each to each, and the rt. ∠ CEH = rt. ∠ GEH (Ax. 10); ∴ the base CH = base GH. (Prop. IV.)

2. In the same way it may be proved that CK = GK.

3. Again, because CH = GH, add to each DH; ∴ CH + DH = GH + DH (Ax. 2) = GD; but GD is < DK + KG (Prop. XX.), and KG = CK; ∴ CH + DH is < CK + DK.

4. In like manner it may be shown that CH + DH is < the sum of any other two lines drawn from C and D to any pt. in AB.

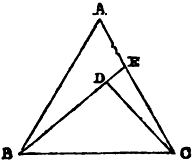
Wherefore, from two gn. pts. C, D, &c.—Q. E. F.

PROP. XXI. THEOR.

GEN. ENUN.—If, from the ends of one side of a triangle, there be drawn two straight lines to a point within the triangle, *these shall be less than the other two sides of the triangle, but shall contain a greater angle.*

PART. ENUN.—Let ABC be a △. From B, C, the extremities of the side BC, draw the st. lines BD, CD, to the pt. D within the △. *Then* BD +

CD are < BA + AC; but the ∠ BDC is > the ∠ BAC.



DEMONST.—1. Produce BD to E; and because two sides of a △ are > the third side, ∴ in △ ABE, BA + AE are > BE. (Prop. XX.) Add to each of these EC; ∴ BA + AC are > BE + EC.

2. Also in the △ CED, DE + EC are > DC. Add to each of these BD; ∴ BE + EC are > than BD + DC; and ∴, *a fortiori*, BA + AC are > BD + DC.

3. Again, because the exterior ∠ of a △ is > the interior opposite ∠ (Prop. XVI.), ∴ the exterior ∠ BDC of △ CDE is > interior opposite ∠ CED; and the exterior ∠ CED of △ ABE is > interior opposite ∠ BAE; ∴, *a fortiori*, ∠ BDC is > the ∠ BAE.

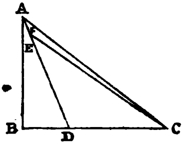
Wherefore, if from the ends &c.—Q. E. D.

It is absolutely essential to the truth of this Theorem that the two lines should be drawn from the *extremities* of one of the sides; for the mere fact that one of the triangles is included within the other is not sufficient. The Proposition is indeed by no means self-evident, as some have supposed; for it will appear from the two subjoined problems that in certain cases the sum of two lines drawn within a △ may either exceed that of the two sides of the △, or contain a smaller ∠.

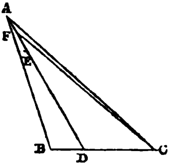
PROP. Q. PROB.

GEN. ENUN.—In a right-angled or obtuse-angled triangle *to determine a point, to which two straight lines being drawn*, one from the extremity of the base farthest from the right angle or obtuse angle, and the other from any point upon the base, *their sum shall exceed that of the two sides of the* triangle.

PART. ENUN.—Let ABC be a △, having the ∠ at B either a rt. ∠ or an obtuse ∠. *Then* it is required to find a pt. within it, to which two lines being drawn from the extremity C, and any pt. D, upon the base BC, their sum shall exceed that of AB + AC.



CONST.—Join AD. Because any two ∠s of a △ are together less than two rt. ∠s (Prop. XVII.); and that the ∠ ABD is either a rt. ∠ or > a rt. ∠ (Hyp.); ∴ the ∠ ADB is < a rt. ∠, and consequently < the ∠ ABD; ∴ AD is > than AB. (Prop. XIX.) Make ED = AB (Prop. III.), and bisect AE in F. (Prop. X.) *Then* F is the pt. required. Join FC.



DEMONST.—1. Because AF = FE, ∴ AF + FC = EF + FC; but AF + FC are > AC (Prop. XX.); ∴ EF + FC are > AC.

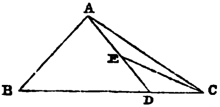
2. Now ED = AB; ∴ by addition, DF + FC are > AB + AC (Ax. 4).

Therefore in the △ ABC, a pt. F has been found such that DF + FC is > AB + AC.—Q. E. F.

PROP. R. PROB.

GEN. ENUN.—From one side of a scalene triangle, *to draw two straight lines to a point within the triangle, which shall contain an angle less than that which is contained by the other two sides of the triangle.*

PART. ENUN.—Let ABC be a scalene △, of which AB is the shortest and BC the longest side. *Then* it is required, &c.



CONST.—From BC cut off DB = AB. (Prop. III.) Join AD: and in AD take any pt. E, and join EC. *Then* the ∠ CED shall be < than ∠ BAC.

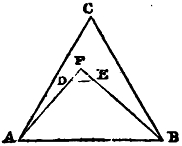
DEMONST.—Since BA = BD, ∴ the ∠ BAD = ∠ BDA (Prop. V.); but the exterior ∠ BDA is > the interior opposite ∠ CED (Prop. XVI.); ∴ also, the ∠ BAD is > the ∠ CED; and ∴, *a fortiori*, ∠ BAC is > the ∠ CED.

Therefore from the base BC of the △ ABC, two lines CE, DE have been drawn to the pt. E, containing the ∠ CED < the ∠ BAC.—Q. E. F.

From Prop. XXI. the following is a deduction:—

PROP. S. THEOR.

GEN. ENUN.—If a trapezium and a triangle are upon the same base, and on the same side of it, and the trapezium falls within the triangle, *the sides of the trapezium are together less than those of the triangle.*



PART. ENUN.—Let the △ ABC and the trapezium ABED be upon the same base AB, and upon the same side of it, and let the trapezium fall within the △. *Then* the sides AD + DE + EB are < AC + CB.

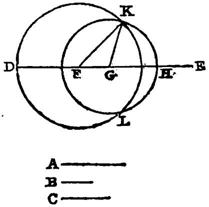
DEMONST.—Produce AD, BE to meet in F. *Then* AF + BF are < AC + BC (Prop. XXI.); and DF + EF are > than DE (Prop. XX.); ∴, *a fortiori*, AD + DE + EB are < than AC + BC.

Wherefore, if a trapezium &c.—Q. E. D.

PROP. XXII. PROB.

GEN. ENUN.—*To make a triangle of which the sides shall be equal to* three given straight lines; but any two whatever of these lines must be greater than the third.

PART. ENUN.—Let A, B, C be the gn. st. lines, of which any two are > the third; viz. A + B > C, A + C > B, and B + C > A. *Then* it is required to make a △ of which the



sides shall be = A, B, C, each to each.

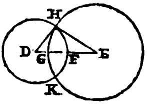
Const.—Take a line DE, terminated towards D, but unlimited towards E, and make DF = A, FG = B, GH = C. (Prop. III.) From cr. F, with rad. FD, describe a ⊙ DKL. From cr. G, with rad. GH, describe a ⊙ HLK (Post. 3). Join KF, KG (Post. 1). *Then* the △ KFG is the triangle required.

Demonst.—By the property of a ⊙ (Def. 15), KF = FD = A; and KG = GH = C; also FG = B. (Const.): ∴ the △ KFG has its three sides, KF, FG, GK = to the three gn. st. lines A, B, C respectively.—Q. E. F.

That the three straight lines must answer the conditions indicated in the enunciation is manifest from Prop. XX.; and, such being the case, the two ⊙^{s} must necessarily cut each other; for since FG + GH are > DF, the ⊙ DKL must meet FE between F and H; and DF + FG being > GH, the ⊙ HLK must meet GD between G and D; so that one of these ⊙^{s} cannot be wholly within the other. Also, because DF + GH is > FG, the two ⊙^{s} cannot lie wholly without each other. It is clear that if DF = GH, the △ will be isosceles, and the Proposition will coincide with Prop. B; and if the three lines be all equal, with Prop. I.

Another mode of solving the Proposition is as follows:—

Taking the longest line A, or DE = A, to form the base of the △, make DF = B, and EG = C. (Prop. III.) From cr. D, with rad. DF, describe the ⊙ FKH. From cr. E, with rad. EG, describe the ⊙ GKH. Join DH, EH. *Then* DEH is the △ required.



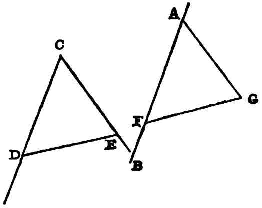
For by property of ⊙, DH = DF = B; EH = EG = C; and DE = A; ∴ the sides of the △ DEH = A, B, C respectively.—Q. E. F.

A similar △ may of course be constructed on the other side of DE.

PROP. XXIII. PROB.

GEN. ENUN.—At a given point in a given straight line, *to make a rectilineal angle equal to* a given rectilineal angle.

PART. ENUN.—Let AB be the gn. st. line; A the gn. pt. in it; DCE the gn. rectilineal ∠. *Then* it is required to make an ∠ at the gn. pt. A = ∠ DCE.



CONST.—Take in CD, CE any pts. D, E, and join DE; make the △ AFG, of which the sides shall be = the st. lines CD, DE, CE, each to each; viz. AF = CD, AG = CE, FG = DE. (Prop. XXII.) *Then* the ∠ FAG is the ∠ required.

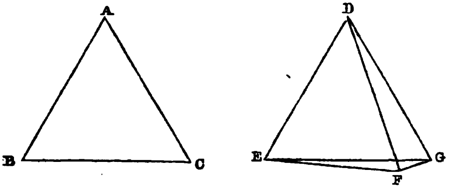
DEMONST.—Because DC, CE = FA, AG, each to each, and the base DE = base FG (Const.), ∴ the ∠ FAG = ∠ DCE. (Prop. VIII.)

Wherefore, at the gn. pt. A, in the gn. st. line AB, the ∠ FAG has been made = the gn. ∠ DCE.—Q. E. F.

PROP. XXIV. THEOR.

GEN. ENUN.—If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other, *the base of that which has the greater angle shall be greater than the base of the other.*

PART. ENUN.—Let ABC, DEF be two triangles, which have the two sides AB, AC = to the two sides DE, DF, each to each, viz. AB = DE,



and AC = DF; but let the ∠ BAC be > the ∠ EDF: *then* the base BC shall be > the base EF.

CONST.—Of the two sides DE, DF, let DE be not > DF; and at the pt. D, in the st. line DE, make ∠ EDG = ∠ BAC. (Prop. XXIII.) Make also DG = AC = DF (Prop. III.), and join EG, GF (Post. 1).

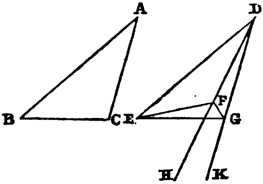
DEMONST.—1. Because AB = DE (Hyp.), and AC = DG (Const.), ∴ the two sides AB, AC = DE, DG, each to each: and the ∠ BAC = ∠ EDG (Const.); ∴ the base BC = base EG. (Prop. IV.)

2. Again, because DG = DF, ∴ ∠ DFG = ∠ DGF (Prop. V.); but the ∠ DGF is > ∠ EGF (Ax. 9); ∴ ∠ DFG is > ∠ EGF; ∴, *a fortiori*, ∠ EFG is > ∠ EGF (Ax. 9).

3. Now the > ∠ of a △ is subtended by its greater side (Prop. XIX.); ∴ in the △ EGF, the side EG is > EF; but EG = BC; ∴ also BC is > EF.

Wherefore, if two △^{s} have two sides &c.—Q. E. D.

There is clearly another case of this Theorem, when DE is > than DF, of which the proof is somewhat different, and is therefore subjoined.



Proceed as before, and produce DF, DG to H, K: *then*, as before, EG = BC; and because DF = DG, the ∠^{s} on the other side of the base of the △ DFG are = (Prop. V.); ∴ the ∠ GFH = ∠ FGK; but ∠ FGK is > ∠ FGE (Ax. 9); ∴ ∠ GFH is > ∠ FGE; ∴, *a fortiori*, ∠ GFE is > ∠ FGE; ∴ side EG is > EF (Prop. XIX.); but EG = BC; ∴ BC is > EF.—Q. E. D.

The reason why Euclid did not adopt this proof may be, that it is, in point of fact, superfluous; for if DE be > DF, DF is < than DE; and by an inverted construction of the figure, the same demonstration will suffice. Of this the student will do well to satisfy himself. That in this case the pt. F falls below the line EG is manifest; for since DG = DF, the pt. G is in the ○^{ce} of a ⊙ described from the cr. D, with rads. DF, and must be above DF, because the ∠ EDG is > than ∠ EDF.

It is also worthy of observation that the conclusion, derived from the inequality of the ∠^{s} contained by the equal sides, is not coextensive with that derived from their equality in Prop. IV. It is not necessary that the area or the remaining ∠^{s} of that which has the greater base should be > than those of the other. Without proceeding further, at least for the present, it will be seen at once that, in the first case, as given in the Proposition, the ∠ ABC will always be <, and, in the second, always > the ∠ DEF.

PROP. XXV. THEOR.

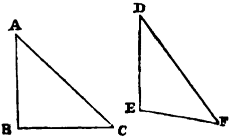
Gen. Enun.—If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, *the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides, equal to them, of the other.*

Part. Enun.—Let ABC, DEF be two $\triangle^{s}$ which have the two sides AB, AC equal to the two sides DE, DF, each to each; *i. e.* AB = DE, and AC = DF: but let the base BC be > the base EF. *Then* the ∠ BAC shall be > the ∠ EDF.

Demonst.—For, if not >, the ∠ BAC is either = or < the ∠ EDF.

1. Now it is not = to it, for then base BC would = base EF (Prop. IV.); but it is not = to it. (Hyp.)

2. Neither is it <, for then BC would be < EF (Prop. XXIV.); but it is not. (Hyp.)



3. ∴ the ∠ BAC is neither = nor < the ∠ EDF; *i. e.* ∠ BAC = ∠ EDF.

Wherefore, if two $\triangle^{s}$ have two sides &c.—Q. E. D.

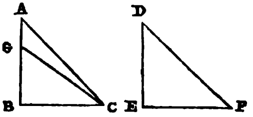
This Proposition is the converse of the preceding.

PROP. XXVI. THEOR.

GEN. ENUN.—If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, *viz.* either the sides adjacent to the equal angles, or the sides opposite to equal angles in each, *then shall the other sides be equal, each to each, and the third angle of the one to the third angle of the other.*

PART. ENUN.—Let ABC, DEF be two $\triangle^s$, which have the $\angle^s$ ABC, BCA = $\angle^s$ DEF, EFD, each to each; viz. ∠ ABC = ∠ DEF, and ∠ BCA = ∠ EFD; also let one side of △ ABC = one side of △ DEF, towards the same parts; *then* the other sides shall be =, each to each, and the third ∠ BAC shall be = third ∠ EDF.

Case 1.—Let the sides adjacent to the = $\angle^s$ in the two $\triangle^s$ be = to each other, *i. e.* let BC = EF; *then,* if AB be not = DE, one of them must be > the other.



CONST.—Let AB be the > of the two; make GB = DE (Prop. III.), and join GC.

DEMONST. *(Ad abs.)*—1. Now because GB = DE, and BC = EF (Hyp.); ∴ in the $\triangle^s$ GBC, DEF, the two sides GB, BC = DE, EF, each to each, and the ∠ GBC = ∠ DEF (Hyp.); ∴ the base GC = base DF, and the △ GBC = △ DEF. (Prop. IV.) Also the remaining $\angle^s$ are =, each to each; ∴ the ∠ GCB = ∠ DFE = ∠ ACB (Hyp.), *i. e.* the less = the greater, which is impossible.

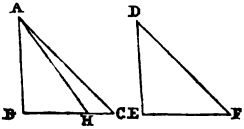
2. ∴ AB is not unequal to DE, *i. e.* AB = DE.

3. Also BC = EF (Hyp.); ∴ the two AB, BC = DE, EF, each to each, and ∠ ABC = ∠ DEF; ∴ the base AC = base DF, and the third ∠ BAC = third ∠ EDF. (Prop. IV.)

Case 2.—Let the sides opposite to = ∠^{s} in each △ be =, *i. e.* let AB = DE. *Then* if BC be not = EF, let BC be the greater of them.

CONST.—Make BH = EF (Prop. III.), and join AH.

DEMONST.—1. Now because BH = EF, and AB = DE (Hyp.); ∴ the two AB, BH = DE, EF, each to each, and the ∠ ABH = ∠ DEF (Hyp.); ∴ the base AH = base DF, and the △ ABH = △ DEF, and the remaining ∠^{s} are also respectively equal (Prop. IV.): ∴ the ∠ AHB = ∠ DFE = ∠ ACB (Hyp.), *i. e.* the exterior ∠ of △ ACH = interior and opposite ∠, which is impossible. (Prop. XVI.)



2. ∴ BC is not unequal to EF, *i. e.* BC = EF.

3. Also AB = DE (Hyp.); ∴ the two AB, BC = DE, EF, each to each, and the ∠ ABC = ∠ DEF (Hyp.); ∴ the base AC = base DF, and the third ∠ BAC = the third ∠ EDF. (Prop. IV.)

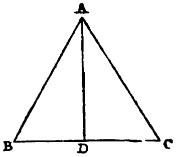
Wherefore, if two △^{s} have &c.—Q. E. D.

This Theorem contains the third and last condition of the equality of two or more Δ^s, and, as in the Fourth and Eighth Propositions, it involves also the equality of the areas and the remaining sides and ∠. For since the two sides AB, BC = DE, EF, each to each, and contain = $\angle^s$, not only does the base AC = base EF, but the Δ ABC = Δ DEF, and the ∠ BAC = ∠ EDF, by Prop. IV. The following are useful deductions:—

PROP. T. THEOR.

Gen. Enun.—If, in an equilateral or isosceles triangle, a straight line be drawn to the base, from the angle contained by the two equal sides, and bisecting the angle; *it shall also bisect the base;* or if it bisect the base, *it shall also bisect the angle;* and if it bisect either the angle or the base, *it shall be perpendicular to the base.*

Part. Enun.—Let ABC be an equilateral or isosceles Δ, having the side AB = AC. From the angular pt. A draw AD to the base BC, and, first, let it bisect the ∠ BAC (Prop. IX.); *then* BC shall also be bisected in D, and AD shall be ⊥ to BC.



Demonst. — Because AB = AC (Hyp.), and AD is common to Δ^s ABD, ACD; ∴ the two sides BA, AD = CA, AD, each to each, and the included ∠ BAD = ∠ CAD (Hyp.); ∴ the base BD = base CD (Prop. IV.); and the ∠ BDA = ∠ CDA, and they are adjacent $\angle^s$; ∴ BC is bisected in D, and the line AD is ⊥ to BC (Def. 10).

Next let BC be bisected in D; *then* also AD shall be ⊥ to BC, and bisect the ∠ BAC.

Demonst.—1. Because BD = CD (Hyp.), and DA is common to Δ^s ADB, ADC; ∴ the two BD, DA = CD, DA, each to each; and the base AB = base AC (Hyp.); ∴ the ∠ BDA = ∠ CDA, and they are adjacent $\angle^s$; ∴ each of them is a rt. ∠.

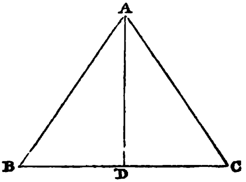
2. Also because the side AB = AC; ∴ the ∠ ABC = ∠ ACB (Prop. V.); ∴ the $\angle^s$ ADB, ABD = ∠ ADC, ACD, each to each; and AD is opposite to = $\angle^s$ in each; ∴ the third ∠ BAD = ∠ CAD (Prop. XXVI.): *i.e.* the ∠ BAC is bisected by AD, which has been shown to be ⊥ to BC.

Wherefore, if in an equilateral &c.—Q. E. D.

PROP. U. THEOR.

Gen. Enun.—If a straight line, drawn ⊥ to the base of a triangle from the opposite angle, bisect either that angle or the base, *the sides containing the angle shall be equal.*

Part. Enun.—Let ABC be a △. Through A draw AD ⊥ to BC (Prop. XII.), and, first, let BC be bisected in D. *Then* the side AB shall be = to AC.



Demonst.—Because BD = CD (Hyp.), and DA is common; ∴, in $\triangle^s$ ADB, ADC, the two BD, DA = CD, DA, each to each, and the rt. ∠ BDA = rt. ∠ CDA (Def. 10); ∴ the base AB = base AC. (Prop. IV.)

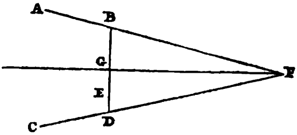
Again, let the ∠ BAC be bisected by the ⊥ AD. *Then,* because the $\angle^s$ BAD, ADB = $\angle^s$ CAD, ADC, each to each, and AD is opposite to = $\angle^s$ in each; ∴ also the base AB = base AC. (Prop. XXVI.)

Wherefore, if a st. line, &c.—Q. E. D.

PROP. V. PROB.

Gen. Enun—Through a given point, *to draw a straight line which shall make equal angles with* two straight lines given in position.

Part. Enun.—Let AB, CD be two st. lines gn. in position; E the gn. pt. *Then* it is required to draw a line through E, which shall make = $\angle^s$ with AB, CD.



Const.—Produce AB, CD to meet in F (Post. 2); bisect the ∠ AFC by the st. line FG (Prop. IX.); through E draw EG ⊥ to FG (Prop. XII.), and produce EG both ways to B and D. *Then* BD is the line required.

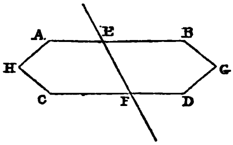
Because the ∠ BFG = ∠ DFG, and the ∠^{s} at G are rt. ∠^{s} (Const.); ∴, in △^{s} BGF, DGF, the ∠^{s} BFG, FGB = ∠^{s} DFG, FGD, each to each, and FG is opposite to the = ∠^{s} in each; ∴ the third ∠ FBG = the third ∠ FDG. (Prop. XXVI.)

Wherefore, through the gn. pt. E, a st. line BD has been drawn making = ∠^{s} with the gn. st. lines AB, CD.—Q. E. F.

PROP. XXVII. THEOR.

GEN. ENUN.—If a straight line, falling upon two other straight lines, makes the adjacent angles equal to one another, *these two straight lines are parallel.*

PART. ENUN.—Let the st. line EF, falling upon the two st. lines AB, CD, make the alternate ∠^{s} AEF, EFD = to one another; *then* AB is ∥ to CD.



CONST. and DEMONST.—1. For, if not, AB and CD, being produced, will meet either towards B, D, or A, C. Let them be produced and meet towards B, D, in the pt. G; *then* the figure EFG is a △, of which the exterior ∠ AEF is > the interior opposite ∠ EFG (Prop. XVI.); but these ∠^{s} are also = (Hyp.), which is impossible; ∴ AB and CD, being produced, do not meet towards B, D.

2. In like manner it may be shown that they do not meet towards A, C.

3. But st. lines which never meet either

way, though produced ever so far, are || to one another (Def. 34); ∴ AB is || to CD.

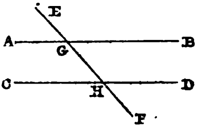
Wherefore, if a st. line, &c.—Q. E. D.

It is clear from the Definition, though Euclid has stated the limitation neither in this nor the following Proposition, that the two lines upon which the other falls should be in the same plane. At the same time, the oversight, if such it be, is so palpable, that it cannot mislead; not to mention that, in the first six books, Euclid treats of plane surfaces only.

PROP. XXVIII. THEOR.

Gen. Enun.—If a straight line, falling upon two other straight lines, makes the exterior angle equal to the interior and opposite angle upon the same side of the line, *or* makes the interior angles upon the same side equal to two right angles, *the two straight lines shall be parallel to one another.*

Part. Enun.—Let the st. line EF, falling upon the two st. lines AB, CD, make the exterior ∠ EGB = the interior and opposite ∠ GHD, upon the same side; or let it make the interior ∠s on the same side, BGH, GHD, together = two rt. ∠s: *then,* in either case, AB is || to CD.



Demonst.—1. Because the ∠ AGH = vertical ∠ EGB (Prop. XV.) = ∠ GHD (Hyp.), and they are alternate ∠s; ∴ AB is || to CD. (Prop. XXVII.)

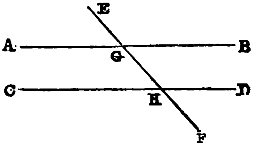
2. Again, because the $\angle^s$ AGH + BGH = two rt. $\angle^s$ (Prop. XIII.) = BGH + GHD (Hyp.); ∴ (by subtraction) the ∠ AGH = ∠ GHD (Ax. 3), and these are alternate $\angle^s$; ∴ AB is ∥ to CD. (Prop. XXVII.)

Wherefore, if a st. line, &c.—Q. E. D.

PROP. XXIX. THEOR.

GEN. ENUN.—If a straight line fall upon two parallel straight lines, *it makes the alternate angles equal to one another; and the exterior equal to the interior and opposite angle upon the same side; and likewise the two interior angles upon the same side equal to two right angles.*

PART. ENUN.—Let the st. line EF meet the ∥ st. lines AB, CD. *Then* it makes the alternate $\angle^s$ AGH, GHD =; also the exterior ∠ EGB = interior and opposite ∠ GHD, upon the same side; and the two interior $\angle^s$ BGH + GHD = two rt. $\angle^s$.



DEMONST.—1. For if the ∠ AGH be not = ∠ GHD, one must be the >.

2. Suppose ∠ AGH > than ∠ GHD, and add to each ∠ BGH; ∴ $\angle^s$ AGH + BGH are > the $\angle^s$ BGH + GHD (Ax. 4); but $\angle^s$ AGH + BGH = two rt. $\angle^s$ (Prop. XIII.); ∴ $\angle^s$ BGH + GHD are < two rt. $\angle^s$.

3. Now st. lines, which, when another line falls upon them, make the interior $\angle^s$ upon the same side $<$ two rt. $\angle^s$, will meet together, if continually produced (Ax. 12); $\therefore$ the st. lines AB, CD will meet, if produced far enough.

4. But they never meet, because they are parallel (Def. 34); $\therefore$ the $\angle$ AGH is not unequal to the $\angle$ GHD, *i. e.* the $\angle$ AGH $=$ $\angle$ GHD.

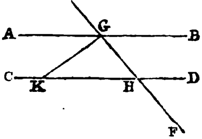
5. Again, the $\angle$ AGH $=$ $\angle$ EGB (Prop. XV.); $\therefore$ $\angle$ EGB $=$ $\angle$ GHD.

6. Add to each the $\angle$ BGH; $\therefore$ $\angle^s$ EGB $+$ BGH $=$ $\angle^s$ BGH $+$ GHD (Ax. 2); but $\angle^s$ EGB $+$ BGH $=$ two rt. $\angle^s$ (Prop. XIII.); $\therefore$ $\angle^s$ BGH $+$ GHD $=$ two rt. $\angle^s$ (Ax. 1).

Wherefore, if a st. line &c.—Q. E. D.

This Proposition is converse to the two which precede. It depends upon the Twelfth Axiom, which, however, so far from being self-evident, is rather a Corollary to the Theorem which it is applied to demonstrate. This circumstance has given much trouble to mathematicians, and involves the doctrine of parallel lines in considerable obscurity. Without adverting to the different means which have been employed to get rid of the difficulty, suffice it to remark that the best way is to seek another solution of this Proposition, from which the proof of the Twelfth Axiom, or, more properly speaking, the Theorem which it involves, will readily be deduced. For this purpose we may proceed thus :—

If the $\angle$ AGH be not $=$ GHD, let AGH be the $>$ of the two. At the pt. G in the st. line EF make the $\angle$ HGK $=$ $\angle$ GHD (Prop. XXIII.), and let GK meet CD in K. *Then* the exterior $\angle$ GHD, of the $\triangle$ KGH, is $>$ than the interior opposite $\angle$ HGK (Prop. XVI.); but (by Construction) the $\angle$ HGK $=$ $\angle$ GHD, the less $=$ the greater,



which is impossible; $\therefore$ the $\angle$ AGH is not unequal to the $\angle$ GHD, *i. e.* the $\angle$ AGH $=$ $\angle$ GHD; but the $\angle$ AGH, &c. &c. &c. as before.

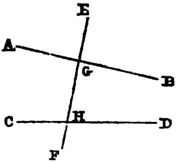
COR.—If a st. line EF meets two st. lines AB, CD, so as to make the two interior $\angle^s$ BGH, GHD taken together less than two rt. $\angle^s$, these st. lines, being continually produced, will at length meet on that side of EF on which are the $\angle^s$ less than two rt. $\angle^s$, *i. e.* towards B and D.

DEMONST.—1. For, if not, AB and CD are either $\parallel$ to one another, or they will meet on the other side of EF.

2. Now they are not $\parallel$, for then the $\angle^s$ BGH $+$ GHD would be $=$ to two rt. $\angle^s$ (Prop. XXIX.); but they are not. (Hyp.)

3. Neither do they meet on the other side of EF; for, if so, the $\angle^s$ AGH, GHC are two $\angle^s$ of a $\triangle$, and $\therefore$ they are $<$ than two rt. $\angle^s$; but the four $\angle^s$ AGH $+$ BGH $+$ GHD $+$ GHC $=$ four rt. $\angle^s$ (Prop. XIII.); and the $\angle^s$ BGH $+$ GHD are $<$ two rt. $\angle^s$ (Hyp.); $\therefore$ the $\angle^s$ AGH $+$ GHC are $>$ two rt. $\angle^s$.

4. $\therefore$ AB and CD do not meet towards A and C: and they are not $\parallel$; $\therefore$ they will meet, if produced, towards B and D.—Q. E. D.

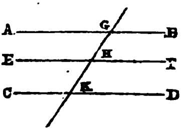


PROP. XXX. THEOR.

GEN. ENUN.—Straight lines, which are parallel to the same straight line, *are parallel to one another.*

PART. ENUN.—Let AB, CD be both $\parallel$ to EF; *then* AB is $\parallel$ to CD.

CONST.—Let the st. line GHK cut AB, EF, CD.



DEMONST.—1. Because AB is $\parallel$ to EF, and GHK cuts them, ∴ the ∠ AGH = alternate ∠ GHF. (Prop. XXIX.)

2. Again, because EF is ∥ to CD, and GK cuts them, ∴ the exterior ∠ GHF = interior ∠ GKD. (Prop. XXIX.)

3. ∴ the ∠ AGK = ∠ GKD (Ax. 1), and they are alternate $\angle^s$; ∴ AB is ∥ to CD.

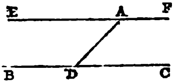
Wherefore, st. lines, &c.—Q. E. D.

It is clear that the same reasoning may be carried on to any number of lines; nor is it necessary that EF should lie between AB and CD; and the student should prove the Theorem, when it is placed either above AB or below CD.

PROP. XXXI. THEOR.

Gen. Enun.—*To draw a straight line through a given point parallel to a given straight line.*

Part. Enun.—Let A be the gn. pt., and BC the gn. st. line. *Then* it is required to draw through A a st. line ∥ to BC.



Const.—In BC take any pt. D, and join AD (Post. 1). At the pt. A, in the st. line AD, make the ∠ DAE = ∠ ADC (Prop. XXIII.); and produce the st. line EA to F (Post. 2). *Then* EF is the line required.

Demonst.—Because the st. line AD, which meets the two st. lines EF, BC, makes the alternate $\angle^s$ EAD, ADC = to one another, ∴ EF is ∥ to BC. (Prop. XXVII.)

Therefore the st. line EAF is drawn through the gn. pt. A ∥ to the gn. st. line BC.—Q. E. F.

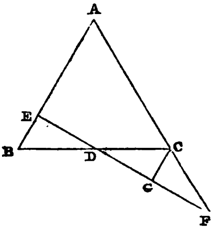
It is clear that the pt. A cannot be so gn. that the line AD shall be in the same st. line with BC.

The following Theorem may now be solved :—

PROP. W. THEOR.

GEN. ENUN.—Of all triangles having the same vertical angle, and whose bases pass through a given point, *the least is that of which the base is bisected in the given point.*

PART. ENUN.—Let BAC be the vertical ∠ of any number of △^{s}, whose bases pass through a gn. pt. D. Let the base BC of the △ ABC be bisected in D; also let AEF be any other △, having the same vertical ∠, and whose base passes through D. The △ ABC is < than the △ AEF.



DEMONST.—1. Through C draw CG ∥ to AB (Prop. XXXI.); then the ∠ EBD = alternate ∠ DCG (Prop. XXIX.); the vertical ∠ EDB = vertical ∠ CDG (Prop. XV.); and the side BD = DC (Hyp.); ∴ △ EBD = △ GCD (Prop. XXVI.), and < than △ CDF.

2. To each add the trapezium ACDE; ∴ the △ ABC is < than △ AEF.

3. The same may be proved with respect to any other △ with the same vertical ∠, whose base passes through D.

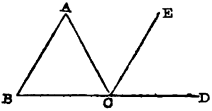
Wherefore, of all triangles &c.—Q. E. D.

PROP. XXXII. THEOR.

GEN. ENUN.—If a side of any triangle be produced, *the exterior angle is equal to the two interior and opposite angles; and* the three interior angles of every triangle *are equal to two right angles.*

Part. Enun.—Let ABC be a △, and let one of its sides BC be produced to D; *then* the exterior ∠ ACD = the interior and opposite $\angle^{s}$ CAB + ABC, and the three interior $\angle^{s}$ ABC + BCA + CAB = two rt. $\angle^{s}$.

Const.—Through the pt. C draw CE ‖ to BA.



Demonst.—1. Because AB is ‖ to CE, and AC meets them, ∴ the ∠ BAC = alternate ∠ ACE. (Prop. XXIX.)

2. Again, because AB is ‖ to CE, and BD meets them, ∴ the exterior ∠ ECD = interior ∠ ABC. (Prop. XXIX.)

3. ∴ the ∠ ACD = $\angle^{s}$ ACE + ECD = $\angle^{s}$ BAC + ABC.

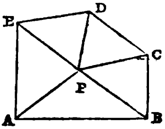
4. Add to each the ∠ ACB; ∴ the $\angle^{s}$ ACD + ACB = $\angle^{s}$ ABC + BCA + CAB (Ax. 2): but the $\angle^{s}$ ACD + ACB = two rt. $\angle^{s}$ (Prop. XIII.); ∴ the $\angle^{s}$ ABC + BCA + CAB = two rt. $\angle^{s}$ (Ax. 1).

Wherefore, if a side of a triangle be produced, &c.—Q. E. D.

Cor. 1.—All the interior $\angle^{s}$ of any rectilineal figure + four rt. $\angle^{s}$ = twice as many rt. $\angle^{s}$ as the figure has sides.

Demonst.—1. For any rectilineal figure ABCDE may be divided into as many $\triangle^{s}$ as the figure has sides, by drawing lines from each of its $\angle^{s}$ to any pt. F within it.

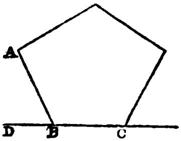
2. Now all the $\angle^{s}$ of these $\triangle^{s}$ = twice as many rt. $\angle^{s}$ as there are $\triangle^{s}$, *i. e.* as there are sides to the figure.



3. And these same $\angle^{s}$ = the $\angle^{s}$ of the figure + $\angle^{s}$ at the pt. F, which is the common vertex of all the $\triangle^{s}$ = $\angle^{s}$ of the figure + four rt. $\angle^{s}$ (2 Cor., Prop. XV.); ∴ all the $\angle^{s}$ of the figure + four rt. $\angle^{s}$ = twice as many rt. $\angle^{s}$ as the figure has sides.—Q. E. D.

COR. 2.—All the exterior $\angle^{s}$ of any rectilineal figure are together = to four rt. $\angle^{s}$.

DEMONST.—Because every interior $\angle$ ABC + its exterior $\angle$ ABD = two rt. $\angle^{s}$ (Prop. XIII.); ∴ all the interior $\angle^{s}$ + all the exterior $\angle^{s}$ = twice as many rt. $\angle^{s}$ as the figure has sides: *i. e.* = all the interior $\angle^{s}$ + four rt. $\angle^{s}$ (Cor. 1); ∴ all the exterior $\angle^{s}$ = four rt. $\angle^{s}$. —Q. E. D.

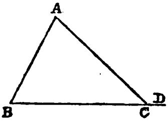


From this Theorem, which is an amplification of the 16th and 17th Propositions, it is clearly deducible that if one of the $\angle^{s}$ of a $\triangle$ be a rt. $\angle$, the sum of the other two is = to a rt. $\angle$; that in an isosceles rt. $\angle^{d}$ $\triangle$, each of the $\angle^{s}$ at the base is half a rt. $\angle$ (Prop. V.); that each of the $\angle^{s}$ of an equilateral $\triangle$ is one-third of two rt. $\angle^{s}$, or two-thirds of a rt. $\angle$; and that if two of the angles of any $\triangle$ are = to two of the angles of another, the third $\angle$ of the one is = to the third $\angle$ of the other. The converse of the Proposition may be thus *partially* enunciated and proved:—

PROP. X. THEOR.

Gen. Enun.—If a straight line be drawn from one of the angles of a triangle, making the exterior = to the two interior and opposite angles, *it shall be in the same straight line with the side of the triangle.*

Part. Enun.—From the angular pt. C of the △ ABC, let a st. line CD be drawn, so that the exterior ∠ ACD = the two interior and opposite $\angle^s$ ABC + BAC; *then* CD is in the same st. line with BC.



Demonst.—Since the ∠ ACD = $\angle^s$ ABC + BAC, add to each side the ∠ ACB; ∴ the $\angle^s$ ACD + ACB = $\angle^s$ ABC + BAC + ACB; but $\angle^s$ ABC + BAC + ACB = two rt. $\angle^s$ (Prop. XXXII.); ∴ $\angle^s$ ACD + ACB = two rt. $\angle^s$ (Ax. 1); ∴ CD is in the same st. line with BC. (Prop. XIV.)

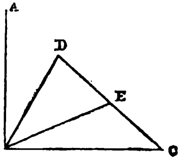
Wherefore, if a st. line &c.—Q. E. D.

To this may be added the following deductions:—

PROP. Y. PROB.

Gen. Enun.—*To trisect* a right angle; that is, *to divide it into three equal parts.*

Part. Enun.—Let ABC be a right ∠; *then* it is required to divide it into 3 = parts.



Const.—In BC take any pt. C, and upon BC describe an equilateral △ BCD (Prop. I.); bisect the ∠ DBC by the st. line BE (Prop. 9); *then* the $\angle^s$ ABD, DBE, EBC are all equal.

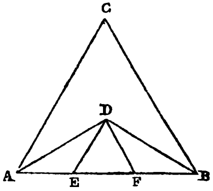
Demonst.—For the ∠ DBC (being one of the $\angle^s$ of an equilateral △) = $\frac{2}{3}$rds of a rt. ∠; ∴ each of the $\angle^s$ DBE, EBC (by construction) = $\frac{1}{3}$rd of a rt. ∠; ∴ also, the ∠ ABD = $\frac{1}{3}$rd of a rt. ∠; ∴ ∠ ABD = ∠ DBE = ∠ EBC.

Wherefore the rt. ∠ ABC has been trisected by the st. lines BD, BE.—Q. E. F.

PROP. Z. PROB.

GEN. ENUN.—*To trisect* a given finite straight line; that is, *to divide it into three equal parts.*

PART. ENUN.—Let AB be the gn. st. line; *then* it is required to divide it into three = parts.



CONST.—Upon AB describe an equilateral Δ ABC (Prop. I.); bisect the $\angle^{s}$ CAB, CBA by the st. lines AD, BD, meeting in D (Prop. IX.); through D draw DE ∥ to AC, and DF ∥ to BC (Prop. XXXI.); *then* AB will be trisected in the pts. E, F.

DEMONST.—1. Because ED is ∥ to AC, the ∠ EDA = alternate ∠ DAC (Prop. XXIX.) = ∠ DAE (Const.); ∴ AE = ED. (Prop. VI.)

2. In like manner, BF = FD.

3. Again, because DE is ∥ to CA, and DF to CB, ∴ exterior ∠ DEF = interior ∠ CAB, and exterior ∠ DFE = interior ∠ CBA (Prop. XXIX.); ∴ the third ∠ EDF of Δ DEF = third ∠ ACB of Δ ABC. (Prop. XXXII.)

4. Hence the Δ DEF is equiangular, and ∴ equilateral (Prop. VI. Cor.); and ∴ AE (= DE) = EF (= FD) = FB.

Therefore the gn. st. line AB has been trisected in the pts. E and F.—Q. E. F.

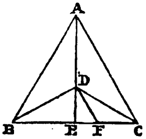
PROP. AA. THEOR.

GEN. ENUN.—If the three angles of a triangle be bisected, and one of the bisecting lines be produced to the opposite side, *the angle contained by this line produced, and one of the others, is equal to the angle contained by the third, and a perpendicular drawn from the common point of intersection of the three lines to the aforesaid side.*

Part. Enun.—Let the three $\angle^s$ of the △ ABC be bisected by the st. lines AD, BD, CD (Prop. IX.); produce AD to E (Post. 2), and from D draw DF ⊥ to BC (Prop. XII.); *then* the ∠ BDE = ∠ CDF.

Demonst.—1. Because the three $\angle^s$ of △ ABC = two rt. $\angle^s$ (Prop. XXXII.), ∴ the $\angle^s$ DBA + DAB + DCF = a rt. ∠ (Const.) = $\angle^s$ DCF + CDF (Prop. XXXII.); ∴, by subtraction, $\angle^s$ DBA + DAB = ∠ CDF (Ax. 3).

2. But the exterior ∠ BDE of △ ABD = $\angle^s$ DBA + DAB (Prop. XXXII.); ∴ the ∠ BDE = ∠ CDF (Ax. 1).

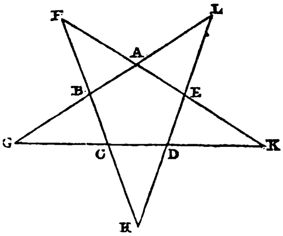


Wherefore, if the three angles &c.—Q. E. D.

PROP. BB. THEOR.

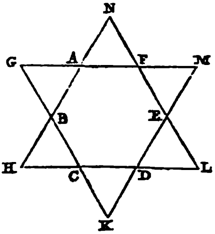
Gen. Enun.—If the sides of an equilateral and equiangular pentagon, or five-sided figure, be produced to meet, *the angles formed by these lines are together equal to two right angles;* and if the sides of an equilateral and equiangular hexagon, or six-sided figure, *be produced to meet, the angles so formed are equal to four right angles.*

Part. Enun.—1. Let ABCDE be an equilateral and equiangular pentagon; produce its sides to meet in F, G, H, K, L; *then* the $\angle^s$ at these pts. together = two rt. $\angle^s$.



Demonst.—For the ∠ ABF (the exterior ∠ of △ LBH) = $\angle^s$ at H and L, and the ∠ BAF (the exterior ∠ of △ GAK) = $\angle^s$ at G and K (Prop. XXXII.); ∴, by addition, $\angle^s$ ABF + BAF = $\angle^s$ G, H, K, L; add to each the $\angle^s$ AFB; ∴ the ∠ at F, G, H, K, L = the three $\angle^s$ of ABF = 2 rt. $\angle^s$. (Prop. XXXII.)

PART. ENUN.—2. Let ABCDEF be an equilateral and equiangular hexagon; produce its sides to meet in G, H, K, L, M, N; the ∠ˢ at these pts. = four rt. ∠ˢ.



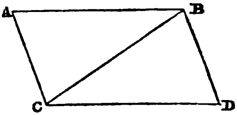
DEMONST.—For they are the ∠ˢ of the two △ˢ GMK, HLN: and the ∠ˢ of any △ = two rt. ∠ˢ (Prop. XXXII.); ∴ the six ∠ˢ at G, H, K, L, M, N = four rt. ∠ˢ.

Wherefore, if the sides &c.—Q. E. D.

PROP. XXXIII. THEOR.

GEN. ENUN.—The straight lines, which join the extremities of two equal and parallel straight lines towards the same parts, *are themselves also equal and parallel.*

PART. ENUN.—Let AB, CD be = and ∥ st. lines, and let them be joined towards the same parts by the st. lines AC, BD; *then* AC is = and ∥ to BD.



CONST.—Join BC.

DEMONST.—1. Because AB = CD, and BC is common to the 2 △ˢ ABC, BCD, ∴ the two sides AB, BC = DC, CB, each to each: and because AB is ∥ to CD, the alternate ∠ ABC = alternate ∠ BCD (Prop. XXIX.); ∴ the base AC = the base BD. (Prop. IV.)

2. Also the ∠ ACB = ∠ CBD.

Hence, because the st. line BC meets the 2 st.

lines AC, BD, and makes the alternate $\angle^{s}$ ACB, CBD = to one another, ∴ AC is ∥ to BD.

And it has been shewn that AC = BD.

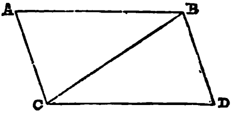
Therefore straight lines &c.—Q. E. D.

PROP. XXXIV. THEOR.

Gen. Enun.—The opposite sides and angles of parallelograms *are equal to one another,* and the diameter *bisects them,* that is, *divides them into two equal parts.*

N. B. A parallelogram is a four-sided figure, of which the opposite sides are parallel: and the diameter is the straight line joining two of its opposite angles.

Part. Enun.—Let ABCD be a $\square^{m}$, and BC its diameter; *then* the opposite sides and $\angle^{s}$ are = to one another, and the diameter BC bisects it.



Demonst.—1. Because AB is ∥ to CD, and BC meets them, ∴ the alternate $\angle$ ABC = alternate $\angle$ BCD (Prop. XXIX.): and because AC is ∥ to BD, and BC meets them, ∴ the alternate $\angle$ ACB = alternate $\angle$ CBD (Prop. XXIX.); ∴ in the $\triangle^{s}$ ABC, CBD, the $\angle^{s}$ ABC, BCA = $\angle^{s}$ BCD, CBD, each to each, and the side BC, adjacent to these = $\angle^{s}$, is common; ∴ the remaining sides are =, each to each, and the third $\angle$ to the third $\angle$ (Prop. XXVI.): *i. e.* AB = CD, AC = BD, and the $\angle$ BAC = $\angle$ BDC.

2. And because the $\angle$ ABC = $\angle$ BCD, and

the $\angle$ CBD $=$ $\angle$ ACB, $\therefore$ the whole $\angle$ ABD $=$ the whole $\angle$ ACD : therefore the opposite sides and $\angle^s$ of $\square^{ms}$ are $=$.

3. Also because AB $=$ CD, and BC is common to the $\triangle^s$ ABC, BCD, $\therefore$ the two AB, BC $=$ DC, CB, each to each, and the included $\angle$ ABC $=$ included $\angle$ BCD; $\therefore$ the $\triangle$ ABC $=$ $\triangle$ BCD (Prop. IV.): *i. e.* the diam. BC bisects the $\square^m$ ABCD.

Wherefore the opposite sides &c.—Q. E. D.

A few important Propositions are deducible from this Theorem, which, after first demonstrating its converse, so far as it is convertible, it may be proper to subjoin.

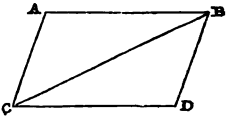
PROP. CC. THEOR.

Gen. Enun.—If the opposite sides, or the opposite angles, of a quadrilateral figure be equal, *the figure is a parallelogram.*

Part. Enun.—Let ABCD be a quadrilateral figure, of which the opposite sides, or opposite $\angle^s$, are $=$; *then* ABCD is a $\square^m$.

Demonst.—First, let the opposite sides be $=$. Join CB.

1. Then, because AB $=$ CD, and BC common to $\triangle^s$ ABC, DBC, $\therefore$ the two AB, BC $=$ DC, CB, each to each, and the base AC $=$ base BD; $\therefore$ the $\angle$ ABC $=$ $\angle$ BCD (Prop. VIII.), and they are alternate $\angle^s$; $\therefore$ AB is $\parallel$ to CD. (Prop. XXVII.)



2. Also the $\angle$ ACB $=$ $\angle$ CBD, and they are alternate $\angle^s$; $\therefore$ also AC is $\parallel$ to BD, and $\therefore$ ABCD is a $\square^m$. (Def.)

3. Again, let the opposite $\angle^s$ be $=$. Now the $\angle$ interior $\angle^s$ of the quadrilateral ABCD $=$ 4 right $\angle^s$ (Prop. XXXII. Cor. 1); also the $\angle^s$ BAC $+$ ACD $=$ $\angle^s$ ABD $+$ BDC

(Hyp.); ∴ the $\angle^s$ BAC + ACD = 2 rt. $\angle^s$, and ∴ AB is ∥ to CD. (Prop. XXVIII.)

4. In the like manner, it may be shewn that AC is ∥ to BD; ∴ ABCD is a $\square^m$.

Wherefore, if the opposite sides &c.—Q. E. D.

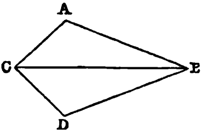
COR.—Hence it appears, from the definitions of the square, and the oblong, the rhombus, and the rhomboid, that these figures are $\square^{ms}$.

That the latter part of the 34th Proposition is not convertible, may be inferred from the following:—

PROP. DD. THEOR.

GEN. ENUN.—A trapezium, of which the adjacent sides are equal, *is bisected by the diameter.*

PART. ENUN.—Let ABCD be a trapezium, of which the sides AB = BD, and CA = CD; *then* the diam. CB bisects it.



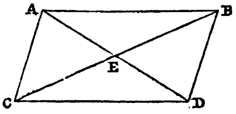
DEMONST.—Because AB = BD, and BC is common to Δ^s ABC, DBC, ∴ the two AB, BC = DB, BC, each to each, and the base AC = base CD; ∴ the Δ ABC = Δ DBC (Prop. VIII. Obs.); and the trapezium is bisected by diam. BC.

Wherefore a trapezium &c.—Q. E. D.

PROP. EE. THEOR.

GEN. ENUN.—The diameters of a parallelogram *bisect each other;* and, *vice versâ,* if the diameters of a quadrilateral figure bisect each other, *the figure is a parallelogram.*

PART. ENUN.—Let ABCD be a parallelogram; its diams. AD, BC bisect each other in E, and *vice versâ.*



DEMONST.—1. Because AC is ∥ to BD, and CB meets them, ∴ the alternate ∠ ACE = alternate ∠ EBD (Prop. XXIX.); and because AC is ∥ to BD, and AD meets them, ∴ the alternate ∠ CAE = alternate ∠ EDB (Prop.

XXIX.) ; ∴ in $\triangle^s$ AEC, BED, the $\angle^s$ ACE, CAE = $\angle^s$ EBD, BDE, each to each, and, adjacent to each, AC = BD ; ∴ AE = ED, and BE = EC (Prop. XXVI.) : therefore the diams. AD, BC, mutually bisect each other.

2. But, let the diams. AD, BC bisect each other in E ; *then*, since, in the $\triangle^s$ AEC, BED, the two AE, EC = the two DE, EB, each to each, and the included vertical ∠ AEC = included vertical ∠ BED (Prop. XV.) ; ∴ the base AC = base BD, and the ∠ ACE = ∠ EBD (Prop. IV.) : ∴ AC is ∥ to BD. In like manner, AB is ∥ to CD ; ∴ ABCD is a $\square^m$.

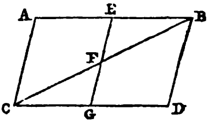
Wherefore the diameter &c.—Q. E. D.

PROP. FF. PROB.

GEN. ENUN.—*To bisect* a parallelogram *by a straight line* drawn from a given point in one of its sides.

PART. ENUN.—Let ABCD be a $\square^m$, E a gn. pt. in the side AB ; *then* it is required to bisect ABCD by a line drawn through E.

CONST.—Draw the diam. BC, and bisect it in F (Prop. X.) ; join EF, and produce it to G (Post. 1 and 2) ; *then* the $\square^m$ ABCD is bisected by EG.



DEMONST.—Because AB is ∥ to CD, and BC meets them ; ∴ the alternate ∠ EBF = alternate ∠ FCG (Prop. XXIX.), and the vertical ∠ EBF = vertical ∠ GCF (Prop. XV.), also BF = FC (Const.) ; ∴ the △ EFB = △ GCF. (Prop. XXVI.)

2. But △ ABC = △ BCD (Prop. XXXIV.) ; ∴, by subtraction, trapezium AEFC = trapezium BFGD (Ax. 3).

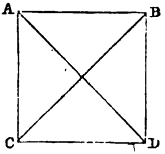
3. Hence, by addition, the trapezium AEGC = trapezium EBDG (Ax. 1).

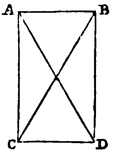
Wherefore the $\square^m$ ABCD has been bisected by the line EG drawn through the gn. pt. E.—Q. E. F.

PROP. GG. THEOR.

GEN. ENUN.—*The diameters* of rectangular parallelograms are *equal to one another;* and in any other parallelogram *the diameter which joins the acute angles is greater than that which joins the obtuse.*

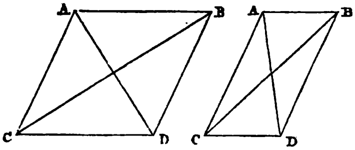
PART. ENUN.—*Case* 1. Let ABCD be a rectangular $\square^m$, as a *square* or *oblong; then* the diam. AD = diam. CB.





DEMONST.—Since, in the $\triangle^s$ ABD, BAC, the two AB, BD = BA, AC, each to each, and the rt. $\angle$ ABD = rt. $\angle$ BAC (Def. 10), ∴ the base AD = the base CB. (Prop. IV.)

PART. ENUN.—*Case* 2. Let ABCD not be rectangular, as a *rhombus* or *rhomboid,* of which the opposite $\angle^s$ ABD, ACD are acute, and CAB, CDB, obtuse; *then* BC is > than AD.



DEMONST.—For, as before, the two AB, BD = BA, AC, each to each; but the $\angle$ BAC is > than $\angle$ ABD, ∴ the base BC is > the base AD. (Prop. XXIV.)

Wherefore the diameters &c.—Q. E. D.

COR.—Since the diams. of a $\square^m$ bisect each other (Prop. EE), if the hypothenuse of a rt. $\angle^d$ $\triangle$ be bisected, the st. line drawn from the rt. $\angle$ to the pt. of bisection is = to half the hypothenuse.

PROP. XXXV. THEOR.

GEN. ENUN.—Parallelograms upon the same base, and between the same parallels, *are equal to one another.*

PART. ENUN.—Let the $\square^{ms}$ ABCD, EBCF (Fig. 2 and 3) be upon the same base BC, and between the same $\parallel^{s}$ AF, BC; *then* the $\square^{m}$ ABCD $=$ $\square^{m}$ EBCF.

Fig. 1.

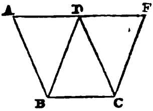


Fig. 2.

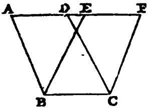
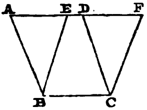


Fig. 3.



DEMONST.—1. If the sides AD, DF of the $\square^{s}$ ABCD, DBCF (Fig. 1), opposite to the base BC, terminate in the same pt. D, it is plain that each of them, being bisected by its diameter (Prop. XXXIV.), $= 2 \triangle$ DBC; $\therefore$ the $\square^{m}$ ABCD $=$ $\square^{m}$ DBCF (Ax. 6).

2. But if the sides AD, EF, opposite to the base BC (Figs. 2 and 3), be not terminated in the same pt.; then, by the property of a $\square^{m}$ AD $=$ BC $=$ EF. (Prop. XXXIV.)

3. To or from each of these equals add or subtract DE; $\therefore$ AD $+$ DE (or AE) $=$ EF $+$ DE (or DF). Also AB $=$ DC.

4. $\therefore$ in the two $\triangle^{s}$ EAB, FDC, the two EA, AB $=$ FD, DC, each to each, and the interior $\angle$ EAB $=$ exterior $\angle$ FDC (Prop. XXIX.); $\therefore$ $\triangle$ EAB $= \triangle$ FDC. (Prop. IV.)

5. Take each of these $\triangle^{s}$ from the trapezium ABCF, and the remainders will be $=$ (Ax. 3): *i. e.* the $\square^{m}$ ABCD $=$ $\square^{m}$ EBCF.

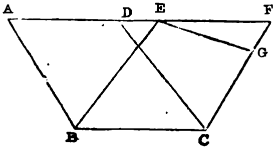
Wherefore $\square^{s.}$ upon the same base, &c.—Q. E. D.

The equality of $\square^{ms}$ is that of their areas; and it appears from this Proposition that any oblique $\square^{m}$ is equal to a rectangular $\square^{m}$, or rectangle, upon the same base and between the same $\parallel^{s}$, *i. e.* of the same base and altitude. Now it is known that the area of a rectangle = the base × its altitude; and ∴ generally the area of any $\square^{m}$ = the base × the ⊥ altitude. The converse of the Proposition is also true.

PROP. HH. THEOR.

Gen. Enun.—Equal parallelograms, upon the same base, *are between the same parallels.*

Part. Enun.—Let the = $\square^{ms}$ ABCD, EBCF be upon the same base BC; *then* they are between the same $\parallel^{s}$: *i. e.* EF is in the same straight line with AD.



Demonst.—1. If not, let EG be in the same st. line with AD; ∴ EG is $\parallel$ to BC, and EBCG is a $\square^{m}$. (Prop. XXXIV.)

2. Now, because the $\square^{ms}$ ABCD, EBCG are upon the same base, and between the same $\parallel^{s}$, they are = to one another.

3. But the $\square^{m}$ ABCD = $\square^{m}$ EBCF; ∴ the $\square^{m}$ EBCG = $\square^{m}$ EBCF, or the < = >, which is impossible.

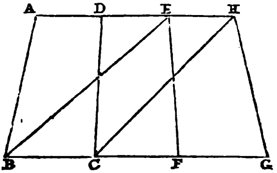
4. ∴ EG is not in the same st. line with AD.

5. In like manner, it may be proved that no other line but EF is in the same st. line with AD; ∴ AD, EF are in the same st. line, and ∴ $\parallel$ with BC. (Prop. XXXIV.)

Wherefore = $\square^{ms}$ &c.—Q. E. D.

PROP. XXXVI. THEOR.

Gen. Enun.—Parallelograms upon equal bases, and between the same parallels, *are equal to one another.*



PART. ENUN.—Let ABCD, EFGH be $\square^{ms}$ upon equal bases BC, FG, and between the same $\parallel^{s}$ AH, BG; *then* the $\square^{m}$ ABCD $=$ $\square^{m}$ EFGH.

CONST.—Join BE, CH.

DEMONST.—1. Because BC $=$ FG, and FG $=$ EH, ∴ BC $=$ EH (Ax. 1).

2. But BC, EH are $\parallel$, and joined towards the same parts by BE, CH; ∴ BE and CH are also $=$ and $\parallel$ (Prop. XXXI.); and ∴ EBCH is a $\square^{m}$. (Prop. XXXIV. Def.)

3. Now, because the $\square^{ms}$ ABCD, EBCH are upon the same base BC, and between the same $\parallel^{s}$ AH, BC, ∴ $\square^{m}$ ABCD $=$ $\square^{m}$ EBCH. (Prop. XXXV.)

4. For the same reason, $\square^{m}$ EFGH $=$ $\square^{m}$ EBCH; ∴ $\square^{m}$ ABCD $=$ $\square^{m}$ EFGH (Ax. 1).

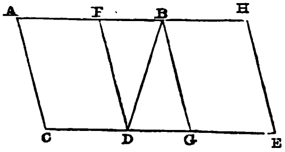
Wherefore $\square^{ms}$ &c.—Q. E. D.

The converse of this Proposition may be demonstrated in the same manner as that of the foregoing; and it is left to the student to shew, that equal $\square^{ms}$, upon $=$ bases in the same straight line, *are between the same* $\parallel^{s}$. (See also Prop. XL.) The following are deductions:—

PROP. KK. THEOR.

GEN. ENUN.—If two sides of a trapezium are parallel, *its area is equal to half that of a parallelogram*, between the same parallels, whose base is equal to the two parallel sides taken together.

Part. Enun. and Const.—Let ABCD be a trapezium, whose side AB is ‖ to CD. Produce CD to E, and make DE = AB. Through D and B draw DF and BG ‖ to AC (Prop. XXXI.); and through E draw EH also ‖ to AC, meeting AB produced in H. *Then* AHCE is a $\square^{m}$ between the same ‖^{s} with the trapezium ABCD, of which the base CE = CD + DE = CD + AB (Const.); and the trapezium ABCD = $\frac{1}{2}$ $\square^{m}$ AHCE.



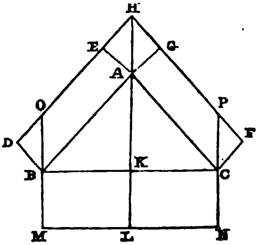
Demonst.—For CD = AF = AB − BF = DE − DG = GE; ∴ the $\square^{m}$ ACDF = $\square^{m}$ BGEH (Prop. XXXVI.); and the △ FBD = △ BDG (Prop. XXXIV.); ∵, by addition, the trapezium ABCD = BDEH (Ax. 2) = $\frac{1}{2}$ $\square^{m}$ ACEH.

Wherefore, if two sides &c.—Q. E. D.

PROP. LI. THEOR.

Gen. Enun.—The sum of the areas of any two parallelograms described on the two sides of a triangle, *is equal to that of a parallelogram on the base*, whose side is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the sides of the former parallelograms produced to meet.

Part. Enun. and Const.—Let ABC be a △, ABDE, ACFG $\square^{ms}$ described upon its sides AB, AC. Produce DE, FG to meet in H. Join HA, and produce it to L, making KL = AH. Through B draw BM ‖ to KL (Prop. XXXI.), and complete the $\square^{m}$ BMNC; *then* the $\square^{m}$ BMNC = $\square^{ms}$ ABDE + ACFG. Produce MB, NC to O and P.



Demonst.—1. Now, because OH is ‖ to BA, and BO to AH, ∴ ABOH is a $\square^{m}$.

2. Also the $\square^m$ ABDE = $\square^m$ ABOH, upon the same base AB, and between the same $\parallel^s$ AB, DH, (Prop. XXXV.) = $\square^m$ BMLK, upon = base KL, and between the same $\parallel^s$ HL, OB. (Prop. XXXVI.)

3. In like manner, $\square^m$ ACFG = $\square^m$ KLNC.

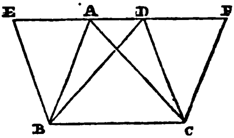
4. $\because$ $\square^{ms}$ ABDE + ACFG = $\square^{ms}$ BMLK + KLNC (Ax. 2) = $\square^m$ BMNC.

Wherefore the sum of the areas &c.—Q. E. D.

PROP. XXXVII. THEOR.

Gen. Enun.—Triangles upon the same base, and between the same parallels, *are equal to one another.*

Part. Enun.—Let the $\triangle$ ABC, DBC be upon the same base BC, and between the same $\parallel^s$ AD, BC; *then* $\triangle$ ABC = $\triangle$ DBC.



Const.—Through B and C draw BE, CF respectively $\parallel$ to CA and BD, and meeting AD produced in E and F; $\therefore$ the figures EBCA, DBCF are $\square^{ms}$. (Prop. XXXIV. Def.)

Demonst.—1. Now the $\square^m$ EBCA = $\square^m$ DBCF upon the same base BC, and between the same $\parallel^s$. (Prop. XXXV.)

2. But the $\triangle$ ABC = $\frac{1}{2}$ $\square^m$ EBCA, because diam. AB bisects it; and the $\triangle$ DBC = $\frac{1}{2}$ $\square^m$ DBCF, because diam. DC bisects it (Prop. XXXIV.); $\therefore$ $\triangle$ ABC = $\triangle$ DBC (Ax. 7).

Wherefore $\triangle^s$ &c.—Q. E. D.

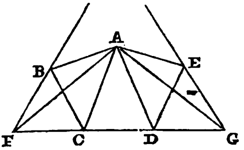
Upon this Theorem depends the following :—

PROP. MM. PROB.

To describe a triangle, which shall be equal to a given pentagon, *and of the same altitude.*

PART. ENUN.—Let ABCDE be the gn. pentagon. It is required to describe a △ of = area, and the same altitude.

CONST.—Join AC, AD. Through B and E draw BF, EG respectively ∥ to AC and AD, and meeting CD produced in F and G. (Prop. XXXI.) Join AF, AG. *Then* AFG is the △ required.



1. Because AC is = to BF, ∴ the △ ABC = △ AFC, upon the same base AC, and between the same ∥ˢ. (Prop. XXXVII.)

2. So the △ AED = △ AGD; ∴ △ˢ ABC + AED = △ˢ AFC + AGD (Ax. 2).

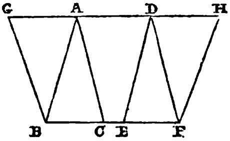
3. To each side add the △ ACD; ∴ the pentagon ABCDE = △ AFG.

Wherefore a △ AFG has been described = to the gn. pentagon ABCDE, and of the same altitude, *viz.* the ⊥ from A upon CD.—Q. E. F.

PROP. XXXVIII. THEOR.

GEN. ENUN.—Triangles upon equal bases, and between the same parallels, *are equal to one another.*

PART. ENUN.—Let the △ˢ ABC, DEF be upon = bases BC, EF, and between the same ∥ˢ BF, AD; *then* △ ABC = △ DEF.



CONST.—Through B and F draw BG, FH respectively ∥ to CA, ED, and meeting AD produced in G and H (Prop. XXXI.); *then* the figures GBCA, DEFH are □^{ms}. (Def.)

DEMONST.—1. Now the □^{m} GBCA = □^{m} DEFH, upon the = base EF, and between the same ||^{s} GH, BF. (Prop. XXXVI.)

2. But the △ ABC = $\frac{1}{2}$ □^{m} GBCA, because diam. AB bisects it; and the △ DEF = $\frac{1}{2}$ □^{m} DEFH, because diam. DF bisects it (Prop. XXXIV.); ∴ △ ABC = △ DEF (Ax. 7).

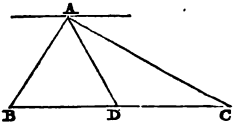
Wherefore △^{s} &c. &c.—Q. E. D.

The following Propositions may be added here :—

PROP. NN. PROB.

GEN. ENUN.—*To bisect* a given triangle *by a line drawn from one of its angles.*

PART. ENUN.—Let ABC be the gn. △. It is required to bisect it by a line drawn from one of its ∠^{s}; for instance, the ∠ BAC.



CONST.—Bisect the opposite side BC in D (Prop. X.), and join AD. The line AD bisects the △ ABC.

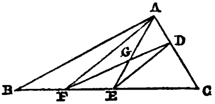
DEMONST.—For the △ ADB = △ ADC, because they are upon = bases BD, CD, and of the same altitude; *i. e.* between the same ||^{s}. (Prop. XXXVIII.)

Wherefore the △ ABC has been bisected by the st. line AD drawn from the ∠ BAC.—Q. E. F.

PROP. OO. PROB.

GEN. ENUN.—*To bisect* a given triangle *by a line drawn from* a given point in one of its sides.

PART. ENUN.—Let ABC be a Δ, D a gn. pt. in the side AC; *then* it is required to bisect the Δ ABC by a line drawn through D.



CONST.—Bisect BC in E (Prop. X.), and join AE, DE; through A draw AF ∥ to DE (Prop. XXXI.), and join DF; *then* DF bisects the Δ ABC.

DEMONST.—1. Because AF is ∥ to DE, the Δ ADE = Δ FDE. (Prop. XXXVII.)

2. From each take the Δ GED; ∴ Δ AGD = Δ FGE (Ax. 3).

3. Also, because EC = EB, the Δ ACE = Δ AEB (Prop. XXXVIII.); ∴, by subtraction, the trapezium DGEC = trapezium ABFG (Ax. 3), and Δ GFE = Δ AGD.

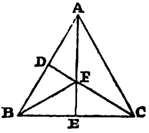
4. ∴, by addition, the Δ DFC = trapezium ABFD (Ax. 2).

Wherefore the Δ ABC has been bisected by the st. line FD drawn through the gn. pt. D.—Q. E. F.

PROP. PP. PROB.

GEN. ENUN.—*To find a point within* a given triangle, *from which lines drawn to the several angles will divide the triangle into three equal parts.*

PART. ENUN.—Let ABC be the gn. Δ. It is required to find a pt. within it, from which lines drawn to the ∠s A, B, C, shall divide it into three = parts.



CONST.—Bisect AB, BC, in D and E. (Prop. X.) Join CD, AE, intersecting in F. *Then* F is the pt. required. Join BF.

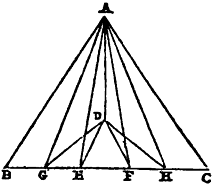
DEMONST.—1. Because AD = DB, ∴ the Δ ADC = Δ BDC, and the Δ ADF = Δ BDF (Prop. XXXVIII.); ∴, by subtraction, Δ AFC = Δ BFC (Ax. 3).

2. Again, because BE = EC, ∴ the Δ AEB = Δ AEC, and the Δ BEF = Δ CEF (Prop. XXXVIII.); ∴, by subtraction, the Δ AFB = Δ AFC = Δ BFC (Ax. 1 and 3).

Wherefore a pt. F has been found within the Δ ABC, from which the lines FA, FB, FC divide the Δ into three = pts.—Q. E. F.

PROP. QQ. PROB.

Gen. Enun.—*To trisect* a given triangle from a given point within it.



Part. Enun.—Let ABC be the gn. △, D the gn. pt. within it. It is required to trisect △ ABC by lines drawn from the pt. D.

Const.—Trisect the side BC in E and F (Prop. Z). Join DE, DF; and from A draw AG, AH respectively ∥ to them. (Prop. XXXI.) Join DG, DH, AD; *then* these three lines will trisect the △ ABC.

Join AE, AF.

Demonst.—1. Because AG is ∥ to DE, the △ ADG = △ AEG, upon the same base AG, and between the same $\parallel^s$. (Prop. XXXVII.)

2. To each add the △ ABG; ∴ the trapezium ADGB = △ AEB (Ax. 2).

3. In like manner, the trapezium ADHC = △ AFC; ∴ by addition, ADGB + ADHC = AEB + AFC (Ax. 2).

4. Subtract each from the △ ABC; ∴ the △ GDH = △ AEF (Ax. 3).

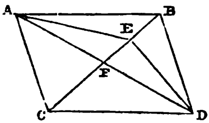
5. Now the △ AEF = △ AEB = △ AFC (Prop. XXXVIII.); ∴ the △ GDH = trapezium ADGB = trapezium ADHC (Ax. 1).

Wherefore the △ ABD has been trisected from a gn. pt. D within it.—Q. E. F.

PROP. RR. THEOR.

Gen. Enun.—If from any point in the diameter of a parallelogram straight lines be drawn to the opposite angles, *they will cut off equal triangles.*

Part. Enun. and Const.—Let ABCD be a $\square^m$, of which



the diam. is BC, E any pt. in BC. Join AE, ED. The △ ACE = △ DCE, and the △ ABE = △ DBE.

Demonst.—1. Because AF = FD (Prop. EE), ∴ the △ AFC = △ DFC, and the △ AFE = △ DFE (Prop. XXXVIII.); ∴, by addition, △ ACE = △ DCE (Ax. 2).

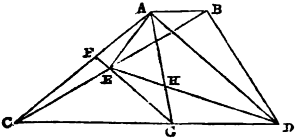
2. But the △ ABC = △ DBC (Prop. XXXIV.); ∴, by subtraction, △ ABE = △ DBE (Ax. 3).

Therefore, if from any pt. &c.—Q. E. D.

PROP. SS. PROB.

Gen. Enun.—*To bisect* a trapezium by a *line drawn from* one of its angles.

Part. Enun. and Const.—Let ABCD be the gn. trapezium, BAC the ∠ from which it is required to be bisected. Draw the diagonals AD, BC. Bisect BC, which is opposite the ∠ BAC, in E. (Prop. X.) Join AE, ED; and through E draw FEG ∥ to AD. (Prop. XXXI.) Join AHG. *Then* AHG bisects the trapezium.



Demonst.—1. Because CE = EB, ∴ the △ AEC = AEB, and the △ DEC = BED (Prop. XXXVIII.); ∴ the figure AEDC = figure AEDB (Ax. 2).

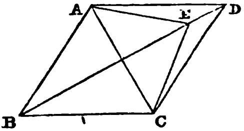
2. Also the △ AEG = △ DEG, upon the same base EG, and between same ∥ˢ EG, AD (Prop. XXXVII.). Take away the common part EHG; ∴ the △ AEH = △ DHG (Ax. 3).

3. Hence the △ ACG = figure AEDC − △ DHG = figure AEDC − △ AEH = figure AEDB − △ AEH = trapezium AGDB.

Wherefore the trapezium ABCD has been bisected by a st. line AG drawn through one of the ∠ˢ BAC.—Q. E. F.

PROP. XXXIX. THEOR.

Part. Enun.—Equal triangles upon the same base, and upon the same side of it, *are between the same parallels.*



Part. Enun.—Let the = $\triangle^s$ ABC, DBC be upon the same base BC, and upon the same side of it; *then* they are between thesame $\|^s$.

Const. and Demonst. (*Ad absurd.*) :—

1. Join AD; and if AD be not $\|$ to BC, through A draw AE $\|$ to BC (Prop. XXXI.), and join EC; *then* the $\triangle$ ABC = $\triangle$ EBC, upon the same base, and between the same $\|^s$. (Prop. XXXVII.)

2. But the $\triangle$ ABC = $\triangle$ DBC (Hyp.); $\therefore$ the $\triangle$ DBC = $\triangle$ EBC, the $>$ = the $<$, which is impossible.

3. $\therefore$ AE is not $\|$ to BC; and, in like manner, no other line except AD is $\|$ to it.

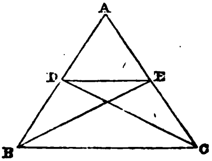
Wherefore = $\triangle^s$ upon &c.—Q. E. D.

Hence the following deduction :—

PROP. TT. THEOR.

Gen. Enun.—If two sides of a triangle be bisected, the straight line which joins the points of bisection *is parallel to the third side.*

Part. Enun.—Let ABC be any $\triangle$; bisect the sides AB, AC, in D and E. (Prop. IX.) Join DE. *Then* DE is ∥ to BC.



DEMONST.—1. Because AD = BD, ∴ △ ADE = △ BDE (Prop. NN).

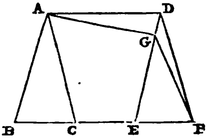
2. In like manner, the △ ADE = △ CED; ∴ the △ BDE = △ CED (Ax. 1.): and they are upon the same base DE; ∴ DE is ∥ to BC, (Prop. XXXIX.)

Wherefore, if two sides &c.—**Q. E. D.**

PROP. XL. THEOR.

GEN. ENUN.—Equal triangles upon equal bases, in the same straight line, and towards the same parts, *are between the same parallels.*

PART. ENUN.—Let the = △ ABC, DEF be upon = bases BC, EF, in the same st. line BF, and towards the same parts; *then* they are between the same ∥ˢ.



CONST. and DEMONST. (*Ad absurd.*) :—

1. Join AD; and if it be not ∥ to BF, through A draw AG ∥ to BF (Prop. XXXI.), and join GF; *then* the △ ABC = △ GEF, upon the = base EF, and between same ∥ˢ. (Prop. XXXVIII.)

2. But the △ ABC = △ DEF (Hyp.); ∴ the △ DEF = △ GEF (Ax. 1), the > = ≮, which is impossible.

2. Hence AG is not $\parallel$ to BF; and, in like manner, no other line except AD is $\parallel$ to it.

Wherefore equal $\triangle^s$ &c.—Q. E. D.

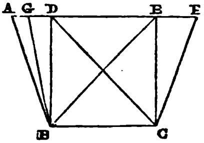
To this may be subjoined the following :—

PROP. UU. THEOR.

Gen. Enun.—Trapeziums upon the same base, having the sides opposite to the base equal, and lying between the same parallels, *are equal to one another;* and equal trapeziums upon the same base, and between the same parallels, *have their sides opposite to the base equal.*

Part. Enun.—1. Let the trapeziums ABCD, EBCF be upon the same base BC, and between the same $\parallel^s$ AF, BC; and let the side AD = side EF. *Then* the trapezium ABCD = trapezium EBCF.

Const.—Join BD, CE.



Demonst. — *Then* the $\triangle$ DBC = $\triangle$ EBC, upon same base BC, and between the same $\parallel^s$ (Prop. XXXVII.); and the $\triangle$ ADB = $\triangle$ FEC, upon the = bases AD, EF, and between same $\parallel^s$. (Prop. XXXVIII.); ∴, by addition, trapezium ABCD = trapezium EBCF.

2. Conversely, let the trapeziums be =, and between the same $\parallel^s$; *then* the side AD = the side EF.

Demonst. *(Ad abs.)*

1. For, since trapezium ABCD = trapezium EBCF, and, as before, the $\triangle$ DBC = $\triangle$ EBC, ∴, by subtraction, $\triangle$ ADB = $\triangle$ CEF.

2. Now, if AD be not = EF, let AD be the >; cut of GD = EF (Prop. III.); and join GB.

3. *Then* the $\triangle$ BDG = $\triangle$ CEF, since they are upon = bases GD, EF, and between same $\parallel^s$. (Prop. XXXVIII.)

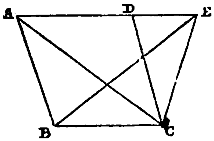
4. But the $\triangle$ ABD = $\triangle$ CEF; ∴ $\triangle$ BDG = $\triangle$ ABD, the < = >, which is impossible.

5. ∴ AD is not unequal to EF, *i. e.* it is = to it.

Wherefore trapeziums &c.—Q. E. D.

PROP. XLI. THEOR.

GEN. ENUN.—If a parallelogram and a triangle be upon the same base, and between the same parallels, *the parallelogram is double of the triangle.*



PART. ENUN.—Let the $\square^{m}$ ABCD and the $\triangle$ EBC be upon the same base BC, and between the same $\|^{s}$ BC, AE; *then* $\square^{m}$ ABCD $= 2 \triangle$ EBC.

CONST.—Join AC.

DEMONST.—Then the $\square^{m}$ ABCD $= 2 \triangle$ ABC, because the diam. AC bisects it (Prop. XXXIV.) $= 2 \triangle$ EBC, upon the same base BC, and between same $\|^{s}$. (Prop. XXXVII.)

Wherefore, if a $\square^{m}$ and a $\triangle$ &c.—Q. E. D.

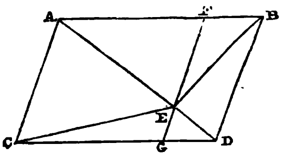
Hence the areas of $\triangle^{s}$ are readily computed; for since the area of a $\square^{m}$ = the base × its $\perp$ altitude, that of a $\triangle = \frac{1}{2}$ base × the $\perp$ altitude.

From this Proposition the following are deductions:—

PROP. VV. THEOR.

GEN. ENUN.—The two triangles formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides, *are together equal to half the parallelogram.*

PART. ENUN.—Let ABCD be a $\square^{m}$, and E any pt. within



it. Draw EA, EB, EC, ED from the pt. E to the extremities of the opposite sides; *then* △ AEC + △ BED, and the △ AEB + CED, are each = to half the $\square^{m}$ ABCD.

Const. — Through E draw FEG ∥ to AC or BD. (Prop. XXXI.)

Demonst.—1. Then the △ AEC = $\frac{1}{2}$ $\square^{m}$ ACGF, and the △ BED = $\frac{1}{2}$ $\square^{m}$ FGDB (Prop. XLI.); ∴ △ AEC + BED = $\frac{1}{2}$ △ ABCD (Ax. 1).

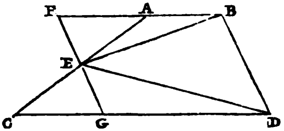
2. Subtract these $=^{s}$ from the whole $\square^{m}$ ABCD; ∴ △ AEB + CED = $\frac{1}{2}$ $\square^{m}$ ABCD.

Wherefore the two $\triangle^{s}$ formed &c.—Q. E. D.

PROP. WW. THEOR.

Gen. Enun.—If two sides of a trapezium be parallel, the triangle contained by either of the other sides, and the two straight lines drawn from its extremities to the bisection of the opposite side, *is equal to half the trapezium.*

Part. Enun.—Let ABCD be a trapezium, having the side AB ∥ to the side CD. Bisect AC in E (Prop. X.), and join EB, ED. *Then* △ BED = $\frac{1}{2}$ trapezium ABCD.



Const. — Through E draw FEG ∥ to BD, meeting CD in G, and BA produced in F. (Prop. XXXI.)

Demonst.—1. Since the alternate ∠ FAE = alternate ∠ ECG (Prop. XXIX.), the vertical ∠ AEF = vertical ∠ CEG (Prop. XV.); and, adjacent to $= \angle^{s}$, the side AE = side EC (Const.); ∴ the △ AEF = △ ECG. (Prop. XXVI.)

2. Add to each the figure AEGDB; ∴ the $\square^{m}$ FGDB = trapezium ABCD.

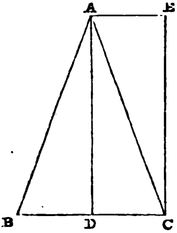
3. Hence the △ BED = $\frac{1}{2}$ $\square^{m}$ FGDB, with same base and altitude (Prop. XLI.) = $\frac{1}{2}$ trapezium ABCD.

Wherefore, if two sides &c.—Q. E. D.

Cor.—Hence a trapezium, which has two of its sides ||, is = to a $\square^{m}$ formed by drawing, through the bisection of one of the sides which are not ||, a line || to the opposite side, and meeting the || sides.

PROP. XX. THEOR.

Gen. Enun.—The perimeter of an isosceles triangle *is greater than* the perimeter of a rectangular parallelogram, which is of the same altitude with, and equal in area to, the given triangle.



Part. Enun. and **Const.** — Let ABC be an isosceles △. Draw AD ⊥ to BC (Prop. XII.), and complete the $\square^{m}$ ADCE, which is of the same altitude with the △ ABC, and has all its $\angle^{s}$ rt. $\angle^{s}$. The $\square^{m}$ ADCE = 2 △ ADC (Prop. XLI.) = △ ABC (Props. T and NN). *Then* the perimeter of the △ ABC is > perimeter of the $\square^{m}$.

Demonst.—1. Because AB = AC, and BD = DC, ∴ the perimeter of △ ABC = 2 AC + 2 DC = 2 (AC + DC); and the perimeter of the $\square^{m}$ ADCE = 2 AD + 2 DC = 2 (AD + DC).

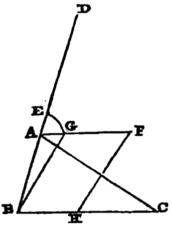
2. But, because ADC is a rt. $\angle^{d}$ △, AC is > AD (Prop. XIX.); ∴ AC + DC is > AD + DC; and ∴ the perimeter of △ ABC is > perimeter of □ ADCE.

Wherefore the perimeter &c.—Q. E. D.

PROP. YY. PROB.

Gen. Enun.—*To describe a parallelogram, of which the area and perimeter shall be respectively equal to* the area and perimeter of a given triangle.

Part. Enun.—Let ABC be the gn. △; *then* it is required to describe a □^{m}, whose area and perimeter shall be respectively = to those of the △ ABC.



CONST.—Produce BA to D, and make AD = AC. (Prop. III.) Bisect BD in E. (Prop. X.) Through A draw AF ∥ to BC. (Prop. XXXI.) With cr. B, and rad. BE, describe a ⊙ cutting AF in G (Post. 3). Join BG; bisect BC in H, and through H draw HF ∥ to BG (Prop. XXXI.); *then* BGFH is the □^{m} required.

DEMONST. 1. Since HF = BG = BE (Def. 15); ∴ HF + BG = 2 BE = BD = BA + AC. (Const.)

2. And since GF = BH = HC (Const.); ∴ GF + BH = 2 HC = BC. (Const.)

3. Hence, by addition, the perimeter of □^{m} BGFH = perimeter of △ ABC.

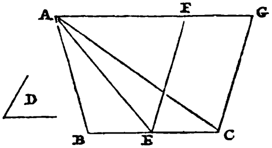
4. Also the □^{m} BGFH = double of a △, upon the base BH, and between the same ∥^{s} (Prop. XLI.), = △ ABC (Prop. NN).

Wherefore a □^{m} has been described, whose area and perimeter = the area and perimeter of the gn. △ ABC.—Q. E. F.

PROP. XLII. PROB.

GEN. ENUN.—*To describe a parallelogram which shall be equal to* a given triangle, *and have one of its angles equal to* a given rectilineal angle.

PART. ENUN.—Let ABC be the gn. △, D the gn. rectilineal ∠; *then* it is required to describe a □^{m} = △ ABC, and having one one of its ∠^{s} = ∠ D.



CONST.—Bisect BC

in E (Prop. X.), and join AE (Post. 1); at the pt. E in the st. line EC make ∠ CEF = ∠ D (Prop. XXIII.); through A draw AFG ∥ to BC, and through C draw CG ∥ to EF (Prop. XXXI.); *then* FECG is the □^{m} required.

Demonst.—1. Because BE = EC, and AG is ∥ to BC, ∴ the △ ABE = △ AEC (Prop. XXXVIII.); ∴ the △ ABC = 2 △ AEC (Ax. 9) = □^{m} FECG. (Prop. XLI.)

2. And the □^{m} FECG has the ∠ CEF = the ∠ D. (Const.)

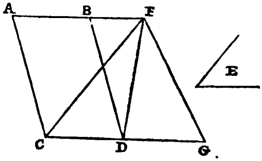
Wherefore a □^{m} has been described = the gn. △ ABC, and having an ∠ = the gn. ∠ D.—Q. E. F.

To this may be added

PROP. ZZ. PROB.

Gen. Enun.—*To describe a triangle which shall be equal to* a given parallelogram, *and having an angle equal to a* given rectilineal angle.

Part. Enun.—Let ABCD be the gn. □^{m}, E the gn. rectilineal ∠; *then* it is required to describe a △ = □^{m} ABCD, and having an ∠ = ∠ E.



Const.—At the pt. C, in the st. line DC, make the ∠ DCF = ∠ E (Prop. XXIII.), and let CF meet AB produced in F; produce CD to G, making DG = CD (Prop. III.), and join FG; *then* FCG is the △ required.

Join FD.

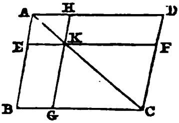
Demonst.—Because CD = DG, ∴ △ FCD = △ FDG (Prop. XXXVIII.); ∴ △ FCG = 2 △ FCD (Ax. 6), = □^{m} ABCD. (Prop. XLI.)

Wherefore a △ FCG has been described, which = the gn. $\square^{\mathrm{m}}$ ABCD, and having the ∠ FCG = gn. ∠ E.—Q. E. F.

PROP. XLIII. THEOR.

GEN. ENUN.—The complements of the parallelograms, which are about the diameter of any parallelogram, *are equal to one another.*

PART. ENUN.—Let ABCD be a $\square^{\mathrm{m}}$, of which the diam. is AC; EH, FG the $\square^{\mathrm{ms}}$ about AC, *i.e. through which* AC *passes;* and BK, KD the other $\square^{\mathrm{ms}}$ which *make up* the whole figure ABCD, and ∴ called *complements. Then* the complement BK = complement KD.



DEMONST.—1. Because the diam. AC bisects the $\square^{\mathrm{m}}$ ABCD (Prop. XXXIV.), ∴ the △ ABC = △ ADC.

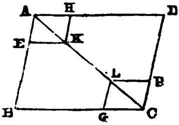
2. For the same reason, the △ AEK = △ AHK, and the △ KGC = △ KFC.

3. ∴ the complement BK = △ ABC − (△ AEK + △ KGC) = △ ADC − (△ AHK + △ KFC) = complement KD.

Wherefore the complements &c.—Q. E. D.

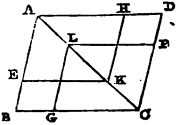
There are other cases of this Proposition; since it is not necessary that the $\square^{\mathrm{ms}}$ about AC should be united at K, or that the complements should be $\square^{\mathrm{ms}}$. The Theorem may be equally proved, when the $\square^{\mathrm{ms}}$ EH, FG are not connected

Fig. 1.



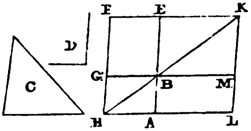
(Fig. 1), and when one of them (Fig. 2) is partly comprised within the other. It will be advisable for the student to demonstrate these cases.

Fig. 2.



PROP. XLIV. PROB.

To a given straight line to *apply a parallelogram, which shall be equal to* a given triangle, *and have one of its angles equal to* a given rectilineal angle.



PART. ENUN.—Let AB be the gn. st. line, C the gn. △, D the gn. rectilineal ∠; *then* it is required to apply to the st. line AB a □^{m} = △ C, and having an ∠ = ∠ D.

CONST. and DEMONST.—Make the □^{m} BEFG = △ C, and having the ∠ EBG = ∠ D (Prop. XLII.), and let BE be in the same st. line with AB; through A draw AH ‖ to BG or EF (Prop. XXXI.), and meeting FG produced in H; join HB.

2. And because the st. line HF falls upon the ‖ st. line AH, EF, ∴ the ∠^{s} AHF + HFE = 2 rt. ∠^{s} (Prop. XXIX.), ∴ the ∠ BHF + HFE are < 2 rt. ∠^{s}; ∴ HB, FE will meet if produced towards B and E. (Prop. XXIX. Cor.)

3. Let them meet in K; through K draw KL

∥ to EA or FH (Prop. XXXI.), and meeting HA produced in L; also produce GB to M.

4. Then HLKF is a $\square^{m}$, of which the diam. is HK; AG, ME are the $\square^{ms}$ about HK, and LB, BF are the complements: ∴ the complement LB = complement BF (Prop. III.) = △ C. (Const.)

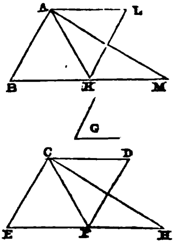
5. Also the ∠ ABM = vertical ∠ EBG (Prop. XV.) = ∠ D. (Const.)

Wherefore the $\square^{m}$ LB has been applied to the gn. st. line AB, which = the gn. △ C, and having the ∠ ABM = gn. ∠ D.—Q. E. D.

The converse Problem may be thus effected.

PROP. AAA. PROB.

Gen. Enun.—To a given straight line *to apply a triangle, which shall be equal* to a given parallelogram, *and having an angle equal to* a given rectilineal angle.



Part. Enun.—Let AB be the gn. st. line, CDEF the gn. $\square^{m}$, G the gn. rectilineal ∠. *Then* it is required to apply to AB a △ = $\square^{m}$ CDEF, and having an ∠ = ∠ G.

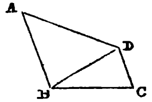
Const.—Draw the diam. CF; produce EF to H, making FH = EF, and join CH. To the st. line AB apply the $\square^{m}$ ABKL = △ CEH, and having the ∠ ABK = ∠ G (Prop. XLIV.); produce BK to M, making KM = BK, and join AM, AK; *then* △ ABM is the △ required.

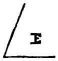
Dmonst.—For the △ ABM = 2 △ ABK (Prop. NN.) = □ BL (Prop. XLI.) = △ CEH (Const.) = 2 △ CEF (Prop. NN.) = $\square^{m}$ CEFD. (Prop. XLI.)

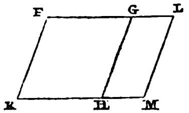
Wherefore to the gn. st. line AB has been applied a △ ABM, which = gn. $\square^{m}$ ED, and has the ∠ ABM = gn. ∠ G.—Q. E. F.

PROP. XLV. PROB.

Gen. Enun.—*To describe a parallelogram equal to* a given rectilineal figure, *and having an angle equal to* a given rectilineal angle.







Part. Enun.—Let ABCD be the gn. rectilineal figure, E the gn. rectilineal ∠; *then* it is required to describe a $\square^{m}$ = ABCD, and having an ∠ = ∠ E.

Const.—Join DB; describe the $\square^{m}$ FH = △ ADB, and having the ∠ FKH = ∠ E (Prop. XLII.); to the st. line GH apply the $\square^{m}$ GM = △ DBC, and having ∠ GHM = ∠ E (Prop. XLIV.); *then* the figure FKML is the $\square^{m}$ required.

Demonst.—1. Since the ∠ FKH = ∠ E = GHM (Const.), add to each side the ∠ KHG, ∴ the $\angle^{s}$ FKH + KHG = $\angle^{s}$ KHG + GHM (Ax. 2).

2. But the $\angle^{s}$ FKH + KHG = 2 rt. $\angle^{s}$ (Prop. XXIX.); ∴ the $\angle^{s}$ KHG + GHM = 2 rt. $\angle^{s}$ (Ax. 1).

3. And because at the pt. H in the st. line GH, the two st. lines KH, HM upon the opposite side of it, makes the adjacent $\angle^{s}$ = 2 rt. $\angle^{s}$; ∴ KH is in the same st. line with HM. (Prop. XIV.)

4. Again, because the st. line GH meets the ∥ st. lines KM, FG, ∴ the alternate ∠ MHG = alternate ∠ HGF. (Prop. XXIX.)

5. To each add the ∠ HGL, ∴ the $\angle^s$ MHG + HGL = $\angle^s$ HGF + HGL (Ax. 2).

6. But the $\angle^s$ MHG + HGL = 2 rt. $\angle^s$ (Prop. XXIX.), ∴ the $\angle^s$ HGF + HGL = 2 rt. ∠ (Ax. 1), and ∴ FG is in the same st. line with GL. (Prop. XIV.)

7. Now, because KF is ∥ to HG, and HG to ML, ∴ KF is ∥ to ML (Prop. XXX.); and KM is also ∥ to FL; ∴ KFLM is a $\square^m$.

8. Hence the figure ABCD = △ ADB + △ DBC = $\square^m$ FH + $\square^m$ HL (Const.) = $\square^m$ KFLM.

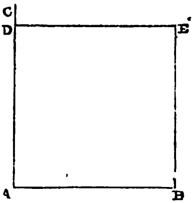
Wherefore a $\square^m$ has been described = to the gn. figure ABCD, and having the ∠ FKM = gn. ∠ E.—Q. E. F.

COR.—From this it is manifest how to a given st. line to apply a $\square^m$, which shall have an ∠ = a gn. rectilineal ∠, and be = a gn. rectilineal figure: viz. by applying to the st. line a $\square^m$ = to the first △ ABD, and having an ∠ = the gn. ∠.

It is also manifest how to proceed, whatever be the number of sides to the gn. rectilineal figure. For it may be divided into as many $\triangle^s$, except two, as the figure has sides; and a $\square^m$ = to each of them in succession is to be applied to the opposite side of each successive $\square^m$.

PROP. XLVI. PROB.

Gen. Enun.—*To describe a square upon a given straight line.*



Part. Enun.—Let AB be the gn. st. line; *then* it is required to describe a square upon it.

Const. and Demonst.—1. From the pt. A draw AC at rt. $\angle^s$ to AB (Prop. XI.); make AD $=$ AB (Prop. III.); through D draw DE $\parallel$ to AB, and through B draw BE $\parallel$ to AD (Prop. XXXI.); $\therefore$ ABED is a $\square^m$; and $\therefore$ AB $=$ DE, and AD $=$ BE. (Prop. XXXIV.)

2. But AB $=$ AD (Const.); $\therefore$ AB $=$ AD $=$ EB $=$ ED (Ax. 1). Hence ABED is equilateral.

3. And because AD meets the $\parallel$ st. lines AB, DE, $\therefore$ the $\angle^s$ BAD $+$ ADE $=$ 2 rt. $\angle^s$ (Prop. XXIX.); but $\angle$ BAD is a rt. $\angle$ (Const.); $\therefore$ also $\angle$ ADE is a rt. $\angle$.

4. Now the opposite $\angle^s$ of $\square^{ms}$ are $=$ (Prop. XXXIV.); $\therefore$ each of the opposite $\angle^s$ ABE, BED is a rt. $\angle$.

5. Hence the figure ABED is equiangular as well as equilateral; it is $\therefore$ a square (Def. 30), and it has been described upon the gn. line AB.—Q. E. F.

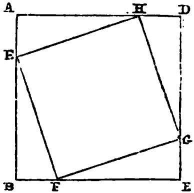
Cor.—Hence every $\square^m$ that has one rt. $\angle$ has all its $\angle^s$ rt. $\angle^s$.

Although not immediately deducible from the above Proposition, the following Theorem may be inserted here.

PROP. BBB. THEOR.

GEN. ENUN.—If in each of the sides of a square, a point be taken at equal distances from the four angles, *the straight lines which join them will also form a square.*

PART. ENUN.—Let ABCD be a square. E, F, G, H, points at = distances from the four $\angle^s$. Join EF, FG, GH, HE. *Then* the figure EFGH is also a square.



DEMONST.—1. Since AE = BF, and AH = EB (Const.); ∴ in the $\triangle^s$ AEH, BEF, the two EA, AH = FB, BE, each to each, and the rt. $\angle$ EAH = rt. $\angle$ EBF; ∴ the base EH = base EF, and the $\angle$ BEF = $\angle$ AHE. (Prop. IV.)

2. In like manner, FG and GH = EF or EH.

3. Hence EF = FG = GH = HE (Ax. 1), and ∴ the figure EFGH is equilateral.

4. Again, the exterior $\angle$ HEB of $\triangle$ AEH = interior $\angle^s$ EAH + AHE (Prop. XXXII.), and the $\angle$ BEF = $\angle$ AHE; ∴, by subtraction, $\angle$ HEF = $\angle$ EAH (Ax. 3) = a rt. $\angle$. (Hyp.)

5. In the same manner the $\angle^s$ at F, G, H may be shewn to be rt. $\angle^s$.

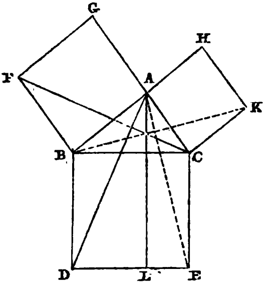
6. Hence the figure EFGH is also equiangular, and ∴ it is a square (Def. 29).

Wherefore, if in each of the sides &c.—Q. E. D.

PROP. XLVII. THEOR.

GEN. ENUN.—In any right-angled triangle, the square which is described upon the side subtending the right angle, *is equal to the squares described upon the sides which contain the right angle.*

PART. ENUN.—Let ABC be a rt. $\angle^{d}$ △, having the rt. ∠ BAC; *then* the square described upon BC = the sum of the squares upon AB and AC (= $AB^2 + AC^2$).



CONST. — On BC describe the squares BDEC, and on AB, AC describe the squares GB, HC (Prop. XLVI.); through A draw AL ∥ to BD or CE (Prop. XXXI.), and join AD, FC (Post. 1).

DEMONST.—1. Now, because each of the $\angle^{s}$ BAC, BAG is a rt. ∠ (Const.) ∴ at the pt. A in the st. line AB, the line CA, AG, on opposite side of it, makes the adjacent $\angle^{s}$ = to two rt. $\angle^{s}$; and ∴ CA is in the same st. line with AG. (Prop. XIV.)

2. For the same reason AB and AH are in the same st. line.

3. Again, because the rt. ∠ DBC = rt. ∠ FBA (Ax. 11), add to each the ∠ ABC; ∴ the ∠ DBA = ∠ FBC.

4. Hence, in the $\triangle^{s}$ ABD, FBC, the two AB, BD = FB, BC, each to each (Const.), and the included ∠ ABD = included ∠ FBC; ∴ the base AD = base FC, and the △ ABD = △ FBC. (Prop. IV.)

5. But the $\square^{m}$ BL = 2 △ ABD, upon the same base BD, and between same $\|^{s}$ AL, BD;

and the square GB $= 2 \triangle$ FBC, upon the same base FB, and between same $\|^s$ FB, GC. (Prop. XLI.)

6. Now the doubles of = things are = to one another (Ax. 6); ∴ the $\square^m$ BL = square GB.

7. In the same manner, it may be proved, by joining AE, KB, that the $\square^m$ CL = square HC.

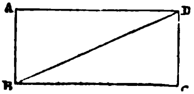
8. Hence, by addition, the square upon BC = sum of the squares upon AB and AC, or $BC^2 = AB^2 + AC^2$.

Wherefore, in any rt. $\angle^d \triangle$, the square &c. —Q. E. D.

This Proposition is no less remarkable for its elegance than for its extensive utility. A few deductions are added for the student's consideration, who will observe, that the side which *subtends* the rt. ∠ in a rt. $\angle^d \triangle$, is called the *hypothenuse*, from the verb *ὑποτείνειν*, *to subtend.*

PROP. CCC. THEOR.

Gen. Enun.—The square described upon the diameter of a rectangular parallelogram, *is equal to the square described upon two of its unequal sides.*



Part. Enun.—Let ABCD be a rectangular $\square^m$, whose diam. is BD; *then* $BD^2 = DC^2 + BC^2$.

Demonst.—For ABD is a rt. $\angle^d$ $\triangle$; ∴ $BD^2 = AB^2 + AD^2 = DC^2 + BC^2$. (Prop. XLVII.)

Wherefore the square &c.

Cor.—If the $\square^m$ be a square, the square upon the diameter is equal to twice the square upon either of the sides; for in this case AB = AD; ∴ $BD^2 = AB^2 + AD^2 = 2\,AB^2$.

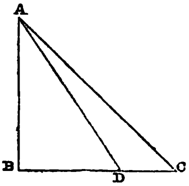
PROP. DDD. THEOR.

Gen. Enun.—If from one of the acute angles of a right-angled triangle, a line be drawn to the opposite side, *the squares of that side and the line so drawn are, together, equal to the squares of the segment adjacent to the right angle and the hypothenuse.*

Part. Enun.—Let ABC be a rt. ∠^{d} △, having the rt. ∠ ABC. From A draw AD to any pt. in the opposite side BC; *then* the squares upon AD and BC = the squares upon AC and BD.

Demonst.—For since $AC^2 = AB^2 + BC^2$ (Prop. XLVII.); add BD^2 to each side; ∴ $AC^2 + BD^2 = AB^2 + BD^2 + BC^2$ (Ax. 2.) $= AD^2 + BC^2$. (Prop. XLVII.)

Wherefore, if from one &c.—Q. E. D.

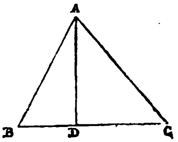


PROP. EEE. THEOR.

Gen. Enun.—In any triangle, if a line be drawn from the vertex perpendicular to the base, *the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.*

Part. Enun.—Let ABC be any △; from the vertex A draw AD ⊥ to the base BC; *then* the difference of the squares upon AB and AC = difference of squares upon BD and DC.

Demonst.—Because ADC is a rt. ∠^{d} △, ∴ $AC^2 = AD^2 + DC^2$; and because ABD is a rt. ∠^{d} △, ∴ $AB^2 = AD^2 + BD^2$; ∴, by subtraction, $AC^2 - AB^2 = DC^2 - BD^2$.

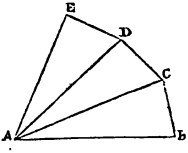


Wherefore in any △, if a line &c.—Q. E. D.

PROP. FFF. PROB.

Gen. Enun.—*To describe a square which shall be equal to* the sum of any number of given squares.

Part. Enun. and **Const.**—Let AB be the side of one of the gn. squares; from B draw BC $\perp^r$ to AB, and = side of second gn. square; join AC, and draw CD $\perp^r$ to AC, and = side of third gn. square; join AD, and draw DE $\perp^r$ to AD, and = side of fourth gn. square; join AE; *then* the square upon AE = sum of squares upon AB, BC, CD, DE.



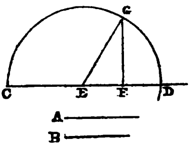
Demonst.—For since the $\angle^s$ ADE, ACD, ABC are all rt. $\angle^s$ (Const.); ∴ (Prop. XLVII.) $AE^2 = AD^2 + DE^2 = AC^2 + CD^2 + DE^2 = AB^2 + BC^2 + CD^2 + DE^2$.

And in the same manner, to any number of squares.—Q. E. F.

PROP. GGG. PROB.

Gen. Enun.—*To describe a square which shall be equal* to the difference of two squares, whose sides are given.

Part. Enun. and **Const.**—Let A, B be the gn. sides of the 2 squares; take a st. line CD terminated at C, but unlimited toward D; make CE = A, and EF = B (Prop. III.); with cr. E, and rad. EC, describe ⊙ CGD (Post. 3); from F draw FG at rt. $\angle^s$ to CD (Prop. XI.); *then* the square described upon FG is the square required.



Join EG.

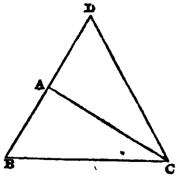
Demonst.—Because EFG is a rt. $\angle$, $EG^2 = EF^2 + GF^2$ (Prop. XLVII.); ∴ $GF^2 = EG^2 - EF^2 = CE^2 - EF^2$ (Def. 15) $= A^2 - B^2$ (Const.)

Hence a square may be described upon GF (Prop. XLVI.) = to the difference of the squares upon A and B.—Q. E. F.

PROP. XLVIII. THEOR.

Gen. Enun.—If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, *the angle contained by these two sides is a right angle.*

Part. Enun.—Let the square upon BC, one of the sides of the △ ABC = the square upon the other two sides AB, AC; *then* ∠ BAC is a rt. ∠.



Const.—From the pt. A draw AD at rt. ∠s to AC (Prop. XI.); make AD = AB (Prop. III.), and join DC.

Demonst.—1. Because DA = AB, ∴ $DA^2 = AB^2$. To each of these add AC^2; ∴ $DA^2 + AC^2 = BA^2 + AC^2$ (Ax. 2).

2. But, because DAC is a rt. ∠, ∴ $DC^2 = DA^2 + AC^2$ (Prop. XLVII.); and $BC^2 = BA^2 + AC^2$ (Hyp.), ∴ $DC^2 = BC^2$, ∴ DC = BC.

3. And because DA = AB, and AC is common to △s CAB, CAD, ∴ the two DA, AC = BA, AC, and the base DC = base BC; ∴ the ∠ BAC = ∠ DAC (Prop. VIII.) = rt. ∠. (Const.)

Therefore, if the square &c.—Q. E. D.

APPENDIX.

IN addition to the numerous Deductions which have been added to the several Propositions of Euclid, it may be advisable to subjoin a few to exercise the ingenuity of the Student. For this purpose, the Enunciation and Construction—in some cases the Enunciation only—has been given; but the Problems involve no difficulty which, with a little consideration, he will not be able to surmount.

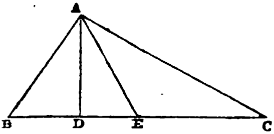
PROP. α. THEOR.—The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles from any point within the figure, except from the intersection of the diagonals.

(Deducible from Prop. XX.)

PROP. β. THEOR.—The sum of the sides of an isosceles triangle is less than the sum of the sides of any other triangle on the same base and between the same parallels.

(Deducible from Prop. XXVII. and Prop. P.)

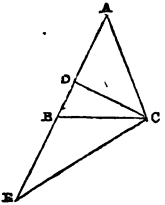
PROP. γ. THEOR.—The difference of the angles at the base of any triangle is double the angle contained by a line drawn from the vertex perpendicular to the base, and another line bisecting the angle at the vertex.



Let ABC be a $\triangle$; draw AD $\perp^r$ to the base BC, and AE bisecting the $\angle$ BAC; then $\angle$ ABC $-$ $\angle$ ACB $=$ 2 $\angle$ DAE.

(Deducible from Prop. XXXII.)

PROP. δ. THEOR.—If from one of the equal angles of an isosceles triangle a line be drawn to the opposite side, and from the same point a line be drawn to the opposite side produced, so that the part intercepted may be equal to the former line, the angle contained by the side of the triangle and the first drawn line is double of the angle contained by the base and the latter.



Let ABC be an isosceles △; draw CD to any point in the opposite side AB; produce AB, making DE = CD, and join CE; *then* the ∠ ACD = 2 ∠ BCE.

(From Prop. XXXII.)

Prop. ε. Theor.—If from one of the equal angles of an isosceles triangle a line, equal to one of the equal sides, be drawn to the opposite side, produced if necessary, the angle formed by this line and the base produced is equal to three times either of the equal angles of the triangle.

(From Prop. XXXII.)

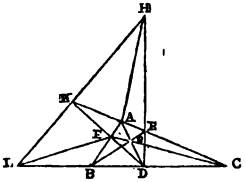
Prop. ζ. Theor.—If the exterior angle of a triangle be bisected, and also one of the interior and opposite angles, the angle contained by the bisecting lines is equal to half the other interior and opposite angle.

(This Theorem, which will explain the principle of Hadley's Sextant, depends upon Prop. XXIX. Cor., and Prop. XXXII.)

Prop. η. Theor.—In any equilateral and equiangular polygon, of which the number of sides is *even*, the opposite sides are respectively parallel.

(From Prop. XXVIII. and XXXII.)

Prop. θ.—If straight lines be drawn from the angles of a triangle, through any point within the triangle, to meet the sides, and the lines joining these points of intersection and the sides of the triangle be produced to meet, the three points of concourse will be in the same straight line.



Let ABC be a △; from the angular pts. draw AD, BE, CF, through any pt. G within the △; join DE, DF, EF, and produce them to meet the sides in H, K, L; join HK, KL; *then* HK, KL are in the same st. line.

(Deducible from Props. XIV. and XXXII.)

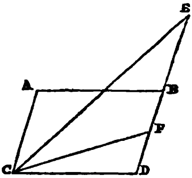
Prop. ι. Theor.—The diameters of a rhombus bisect each other at right angles.

Prop. κ. Prob.—To inscribe a square in a gn. rhombus.

(By means of Prop. XXXII.)

Prop. λ. Prob.—From one of the angles of a parallelogram to draw a line to the opposite side, which shall be equal to that side, together with the segment of it which is intercepted between the line and the opposite angle.

Let ABCD be the $\square^m$; ACD the ∠ from which the line is to be drawn; produce DB to E, and make BE = BD; join CE, and at the pt. C in the st. line EC make ∠ ECF = ∠ CEF; then CF is the line required.



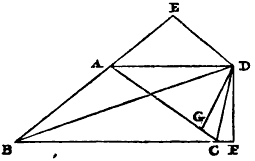
Prop. μ. Theor.—The triangle contained by the straight lines joining the points of bisection of the three sides of a given triangle, is one-fourth part of the given triangle, and is equiangular with it.

(Deducible from Prop. XLI.)

Prop. ν. Theor.—If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to either of the two equal sides be each a right angle, the triangles shall be equal and similar to each other.

(Deducible from Prop. XLVII.)

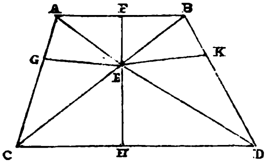
Prop. ο. Theor.—If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines a line be drawn to the opposite angle of the triangle, it will bisect that angle.



Let ABC be a Δ; bisect the exterior $\angle^s$ EAC, ACF in AD, CD, meeting in D, and join DB: the ∠ ABC is bisected in BD. Drop the $\perp^{rs}$ DE, DF, DG.

(Depending on Prop. XIV. XXVI. XLVII.)

Prop. π. Theor.—If from any point within or without any rectilinear figure perpendiculars be let fall on every side, the sum of the squares of the alternate segments made by them will be equal.



Let ABCD be the figure; E the given pt.; drop the $\perp^s$ EF, EG, EH, EK; *then* $AF^2 + BK^2 + DH^2 + CG^2 = FB^2 + KD^2 + HC^2 + GA^2$, and so for any number of sides.

(Deducible from Prop. XLVII.)

Prop. ρ. Theor.—If, in the figure to Prop. XLVII., CF, BK be joined, they will intersect AL, the perpendicular to BC, in the same point.

(See Prop. F.)

Prop. σ. Theor.—If, in the same figure, perpendiculars be drawn from the points F, K, to BC produced, the line intercepted between B and the perpendicular from F will be equal to the line intercepted between C and the perpendicular from K; and the sum of these lines will be equal to BC.

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